9 My Life On The 2.71828. . . List

Important Stuff

PROBLEM

Here's a recurrence with an unknown starting point:

$$W(n) = \begin{cases} S, & n = 0 \\ \frac{1}{3}W(n-1) + 6, & n > 0 \end{cases}$$

1. Let $S = 9$. Calculate $W(0)$ through $W(4)$.
2. Let $S = 36$. Calculate $W(0)$ through $W(4)$ and plot them on a number line.
3. Let $S = -18$. Calculate $W(0)$ through $W(4)$ and plot them on the *same* number line. What do you notice?
4. Test out this closed rule for $W(n)$:

$$W(n) = (S - 9) \cdot \left(\frac{1}{3}\right)^n + 9$$

What happens when $S = 9$? When $S > 9$? When $S < 9$?

$S = 9$ is like the 1919 World Series, or a sad puppy.

1. Here's a recurrence with an unknown starting point:

$$M(n) = \begin{cases} S, & n = 0 \\ 3M(n-1) - 18, & n > 0 \end{cases}$$

(a) Let $S = 9$. Calculate $M(0)$ through $M(4)$.
(b) Let $S = 10$. Calculate $M(0)$ through $M(4)$.
(c) Let $S = 8$. Calculate $M(0)$ through $M(4)$.
(d) Determine a closed rule for $M(n)$, and check it.

Check before you wreck $Y(0)$-self, of course.

2. Consider the two-step recursive rule

$$K(n) = 14K(n-1) - 33K(n-2)$$

(k) Write down everything you've learned about this type of recursive rule in the course.
(a) Find a closed rule for $K(n)$ for the starting values $K(0) = 2$ and $K(1) = 14$.
(r) Find a closed rule for $K(n)$ for the starting values $K(0) = 1$ and $K(1) = 0$.

A famous Boston Celtic wore #33. Everybody's heard about him.

(e) Find a closed rule for $K(n)$ for the starting values $K(0) = 0$ and $K(1) = 1$.

(n) Find a closed rule for $K(n)$ for the starting values $K(0) = 17$ and $K(1) = 42$.

Use parts (r) and (e). . .

3. At noon, Cal is caffeine free, but he'll fix that by chugging a 16.9-ounce bottle of Coke Zero at the end of every hour for all eternity. Each bottle of Coke Zero contains C mg of caffeine. Like Ben, 10% of Cal's caffeine is metabolized each hour. Let's measure his caffeine level at the top of each hour starting at noon.

(t) How much caffeine is in Cal at 6 pm? Give your answer in terms of C.

(e) How much caffeine is in Cal at midnight? Give your answer in terms of C, and maybe clean it up a little.

(r) How much caffeine is in Cal after 1000 hours? Say, about how big is $(0.9)^{1000}$ anyway?

(i) There's 48 mg of caffeine in a 16.9-ounce Coke Zero. In the long run, how much caffeine will run through Cal if he keeps this up?

Hey! That's a 250mL bottle, I'll have you know. . .

At noon, Cal's caffiene level is (Coke) 0. At 1 pm, Cal's caffeine level is C. At 2 pm, Cal's caffeine level is $0.9C + C$. You'll be glad you wrote it that way and not as $1.9C$.

4. Find a closed rule that agrees with this recursive rule. If you like, you can make one of them cobwebby thingies.

$$f(n) = \begin{cases} 8, & n = 0 \\ \frac{1}{2} f(n-1), & n > 0 \end{cases}$$

5. Find a closed rule that agrees with this recursive rule.

Maybe another cobwebby thingy too?

$$g(n) = \begin{cases} 18, & n = 0 \\ \frac{1}{2} g(n-1) + 5, & n > 0 \end{cases}$$

6. Let $g(n) = \frac{1}{2}g(n-1)+5$. Now let $f(n) = g(n)-10$. . . then also $f(n-1) = g(n-1) - 10$.

(a) Substitute to show that $f(n) = \frac{1}{2}f(n-1)$.

(b) How does this relate to the last two problems?

(c) From whence did we yoink this magic 10?

Neat Stuff

7. Let $e(n)$ be defined by this recursive rule:

$$e(n) = \begin{cases} 1, & n = 0 \\ e(n-1) + \frac{1}{n!}, & n > 0 \end{cases}$$

Calculate $e(20)$ to as many decimal places as you care to.

8. Let $Y(n)$ be defined by this recursive rule:

$$Y(n) = \begin{cases} 60000, & n = 0 \\ 1.005Y(n-1) - 500, & n > 0 \end{cases}$$

(a) Find the solution to $Y = 1.005Y - 500$. Call this number F. This is the *fixed point* for the recurrence given.
(b) Calculate $Y(0)$. How far is it from the fixed point?
(c) Calculate $Y(1)$. How far is it from the fixed point?
(d) Calculate $Y(2)$. How far is it from the fixed point? Find a pattern.
(e) Let $Z(n) = Y(n) - F$. Write a simple recursive rule for $Z(n)$.
(f) Write a closed rule for $Z(n)$.
(g) How far away will $Y(n)$ be from the fixed point F? Use this to write a closed rule for $Y(n)$.

If you got a problem, $Y(0)$ I'll solve it. Check out the hook while the DJ revolves it...

$Z(n-1) = Y(n-1) - F$ too.

9. Monica buys a Prius valued at \$22,800. She gets a 4.8% APR loan and plans to pay \$500 per month for 48 months.
(a) How much money will Monica still owe at the end of the 48 months?
(b) There is a more expensive car for which a \$500 per month payment would be a fixed point, an *interest-only loan*. How much would that car cost?
(c) How far away from the fixed point is \$22,800?
(d) Determine how far away from the fixed point Monica will be after the 48 months, using the method of problem 8.
(e) How much money will Monica still owe at the end of the 48 months?

Another convenient percentage, how nice. The "fixed point" car would be one where the \$500 payment exactly equals the amount of interest in the month. A *really* expensive car!

10. Repeat problem 9, but replace the monthly payment by the variable p. Use this to determine the monthly payment that Monica should make to leave a \$0 balance at the end of 48 months.

This choice of variable reminds us to tell you that the Yellow Snow Ice Cream place is really good! Just past Grub Steak.

11. Here's a two-term recurrence with a shocking twist:

$$J(n) = \begin{cases} 4, & n = 0 \\ 3, & n = 1 \\ 7J(n-1) - 10J(n-2) + 8, & n > 1 \end{cases}$$

The "+8" wrecks it, or maybe not. Use fixed-point analysis to find a closed rule for $J(n)$.

Almost as shocking as that Rear Admiral who claimed to be the first to fly over the North Pole but was later shown not to have made it. Don't you know about him?

12. Find a closed rule for this recursive rule.

$$K(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ 2J(n-1) + 3J(n-2) + 100, & n > 1 \end{cases}$$

13. Consider the two-step recursive rule

$$L(n) = 4L(n-1) - 4L(n-2), \qquad n > 1$$

(j) Find a closed rule for $L(n)$ if $L(0) = 1$ and $L(1) = 2$.
(u) Find a closed rule for $L(n)$ if $L(0) = 0$ and $L(1) = 2$.
(d) Find a closed rule for $L(n)$ if $L(0) = 1$ and $L(1) = 4$.
(i) Find a closed rule for $L(n)$ if $L(0) = 1$ and $L(1) = 0$.
(t) Find a closed rule for $L(n)$ if $L(0) = 0$ and $L(1) = 1$.
(h) Find all scaled points for $(x, y) \mapsto (y, -4x + 4y)$.

14. For each recursion on points, find all the *scaled points*, points where (x, y) maps to a multiple of itself (kx, ky).

(j) $(x, y) \mapsto (x, -y)$
(a) $(x, y) \mapsto (y, -x)$
(i) $(x, y) \mapsto (3x, 3y)$
(m) $(x, y) \mapsto \left(\frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} + \frac{y}{2}\right)$
(e) $(x, y) \mapsto \left(\frac{3}{5}x - \frac{4}{5}y, \frac{4}{5}x + \frac{3}{5}y\right)$

A lot of these rules turn points, rotating. It reminds us of that band that sang that song about turning, turning, turning. Surely you've heard of them.

Tough Stuff

15. Here's a fun nonlinear recurrence.

We know, they're all fun. But this one's *especially* fun!

$$\varepsilon(n) = \begin{cases} 1, & n = 0 \\ \varepsilon(n-1) \cdot (2 - \ln \varepsilon(n-1)), & n > 1 \end{cases}$$

What is $\varepsilon(1)$? What is $\varepsilon(5)$? What's going on?

16. Let $T(n)$ be defined as in the past two days by this recursive rule.

$$T(n) = \begin{cases} 0, & n = 0 \\ nT(n-1) + n, & n > 1 \end{cases}$$

Calculate the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{T(n)}\right)$$

That giant Π either stands for "product", or it's one of those big ships from TRON. End of line.