15 *log(Goodbye)*

Problems are categorized by the week they review. Skip around as much as you like. If you'd like to make a short presentation on something you learned in this course, let us know and we can take a camera shot.

What Madonna song are we referring to in this title?

Something you learned *in this course*, not at karaoke. There, we learned that Ellie is a fantastic singer and Just Dancer, and that somehow "King Tut" is still available in karaoke machines. (Well sung, Andy.)

Week 1 Stuff

10ooh.

$$OOH(EE + (AH)^2) + T\left[(I + A)NG\right] +$$

$$(WALLA)^2 + B\left[(I + A)NG\right] = ?$$

6+8. Let

$$F(n) = \begin{bmatrix} 0 & 1 \\ -12 & 7 \end{bmatrix}^n \begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Astoundingly, both tables 6 and 8 made the same problem. Coincidence? Or cheating?! Only their hairdresser knows for sure.

(a) Let $a(n)$ be the first number in the vector $F(n)$. Write a recursive rule for $a(n)$, like

$$a(n) = \begin{cases} \boxed{}, & n = 0 \\ \boxed{}, & n = 1 \\ \boxed{\quad \textit{some rule} \quad}, & n > 1 \end{cases}$$

(b) Write a closed rule for $a(n)$ that does not involve matrices.

12. Find a rule for the sum of the first n Fibonacci numbers, starting with $F(0) = 0$.

This doesn't have to use the closed form. Look for patterns.

12. Find a rule for the sum of the first n powers of 3, starting with $3^0 = 1$.

6. Let

$$h(n) = \begin{cases} 4, & n = 0 \\ -8, & n = 1 \\ -6h(n-1) - 5h(n-2), & n > 1 \end{cases}$$

(a) Find $h(0)$ through $h(5)$.
(b) Find a closed rule for $h(n)$.

Ooh, I found $h(0)$! It's like searching for Waldo except more boring.

8. Let $f(x) = 2x + 1$.

(a) Determine $f(f(x))$ and $f(f(f(x)))$.

(b) Let $f^n(x)$ be the same as $f(x)$ iterated n times. Keep iterating $f(x)$ until you can find a rule for $f^n(x)$ in terms of n.

Note: this is not pronounced "ffffffx".

12. Find the value of

$$\frac{1}{\phi} + \frac{1}{\phi^2}$$

where ϕ is the golden ratio $\dfrac{1+\sqrt{5}}{2}$.

Week 2 Stuff

12. Consider parallelogram $BASE$ with vertices at $(1,1)$, $(6,3)$, $(8,7)$, and $(3,5)$.

(a) Determine A_{BASE}, the area of the parallelogram.

(b) The parallelogram is transformed according to rule $S : (a,b) \mapsto (b, a+b)$. Find the new coordinates and draw the new parallelogram.

(c) If this transformation is repeated x times, what happens to the shape and orientation of the parallelogram?

(d) What happens to $A_{BASE \cdot S^x}$, the area of the parallelogram after x transformations?

All that we want is to find the area. You're gone tomorrow. Boy.

It appears to be getting little in the middle. Our apologies for the tough notation here, but we hope you'll see the sign.

5. A New Yorker came to a party with 0.8 oz of apple sauce in his body. He then guzzled 1.2 oz of apple sauce every hour. His body processes 8% of the apple sauce each hour.

(z) Write a recursive rule for the amount of apple sauce in the New Yorker's body after n hours.

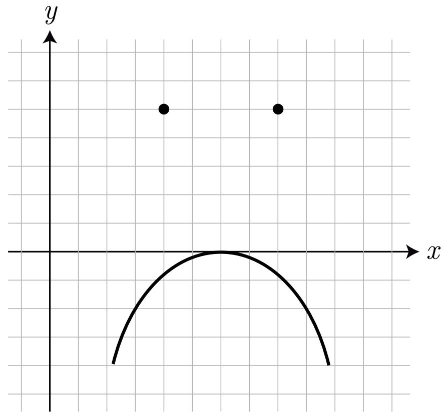
(a) Determine the amount of apple sauce in his body after 3 hours.

(c) Find a closed rule for the amount of apple sauce in his body after n hours.

(h) How much apple sauce should he consume each hour to maintain a fixed point of 2.5 oz of apple-sauciness?

7. An evil undergrad drew the frowning face below:

Maybe it was one of Ramona's exes? Clearly the effect of too much infinity norm and not enough apple sauce.



Help turn that frown upside down! Find a transformation in the form $(x, y) \mapsto$ something clever that turns the picture into a smiley face!

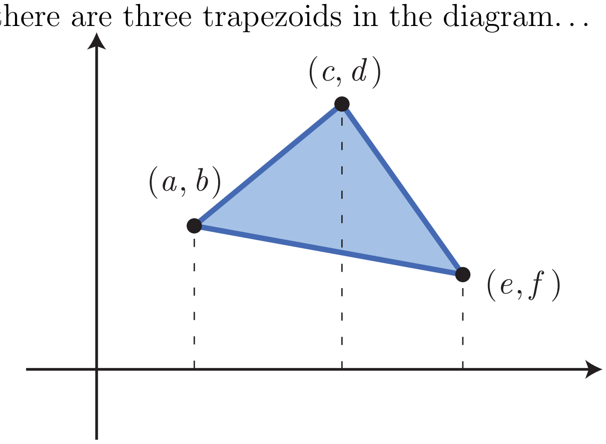
Can you think of something clever? Absolutely!

8. Use this diagram to find a rule for the area of a triangle with vertices (a, b), (c, d), and (e, f). By the way, the area of a trapezoid is given by

But wait! What if one of the trapezoids is a parallelogram instead! OH NOES!

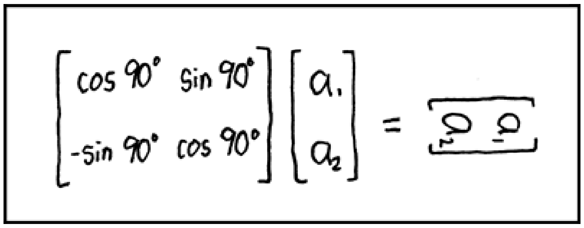
$$A = \frac{1}{2}(b_1 + b_2)h$$

and there are three trapezoids in the diagram...



Week 3 Stuff

12. Explain this XKCD cartoon. Note: $\cos 90° = 0$ and $\sin 90° = 1$.



10ee. Let

$$B\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \quad A\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad N\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}, \text{ and } \quad G\begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

(r) Plot points and connect (recommend using a $\frac{1}{3}$-unit scale) to form $BANG$.

BANG!

(i) Multiply each point in $BANG$ with the matrix

$$\begin{bmatrix} 3 & -3 \\ 1 & 1 \end{bmatrix}$$

to get big-$BANG$. Draw big-$BANG$ and $BANG$ on the same axes.

BANG!!!!!!

(n) Find the areas of $BANG$ and big-$BANG$ geometrically. How does this relate to the matrix above?

(a) Multiply the same matrix with big-$BANG$ to make big-whopping-$BANG$.

BANG!!!!!!!!!!!!!!!!!!!!!!!

($\star$) Find scaled vectors of the matrix and scaling factors for each.

(?) What is the sum of your scaling factors? What is the product of your scaling factors? Notice anything? Go back and look at some other matrices and see if you find the same result.

This is a truly outstanding question and well worth exploring. Thanks!

12. Take any of the shapes used in this week's openers and transform them under the matrix

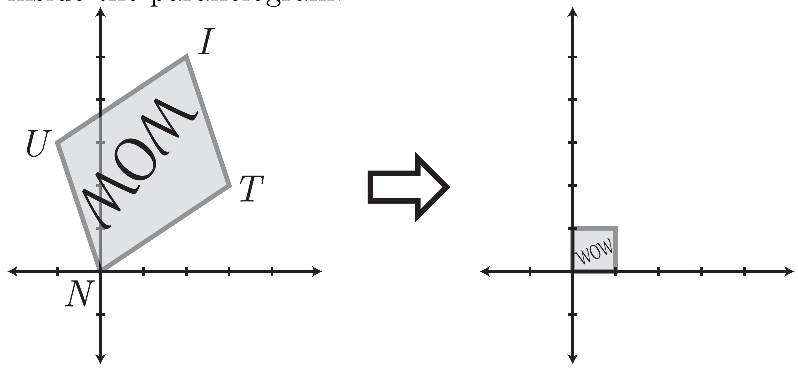
This matrix makes things flatter than that lady singing about her brand new whatever it was. . . roller skates? No idea.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

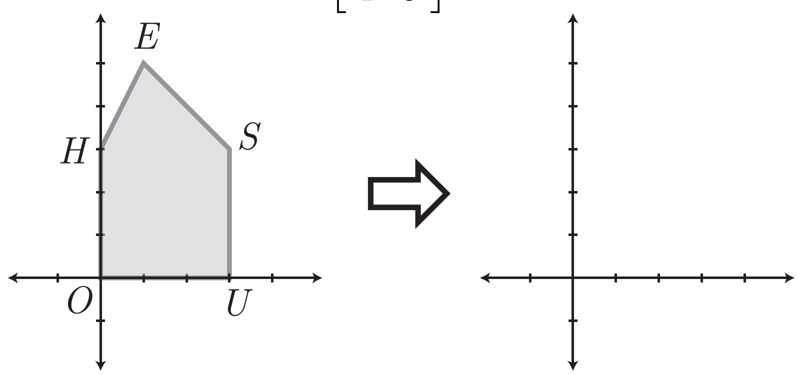
(a) What happens?

(b) When, in general, does this sort of thing happen?

9. Find a transformation that makes this parallelogram into a unit square. Take note of the orientation of the stuff inside the parallelogram.



9. What happens when the *HOUSE* below goes under renovation with the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$?



5. The set designer for Madonna's next Dar-Wen tour is laying out the stage for her performance of "Like a Virgin" at the Vatican. The mattress is aligned on a coordinate grid with the vertices at T $(3, 3)$, O $(0, 0)$, U $(-6, 6)$ and R $(-3, 9)$. At one point in the performance the mattress will transform according to matrix

$$M = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

Graph the original and the new mattress con-$TOUR$ and describe the effect. What will it look like?

This is to follow her "Pope-a don't Preach" number. (Apparently she'd been on the Bo-ryll tour last year.)

This is all part of Madonna's controversial new "Attempted Chocolate Suicide" stage show.

3. (a) Plot the point $X = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

(b) Let

$$T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}.$$

Find TX, the location of the new point after X is transformed by T, and plot it.

(c) Apply the transformation four more times and plot these points. What do you notice?

Circle gets the square?

(d) In what quadrant is $T^{25}X$? Try to figure this out without determining its coordinates.

1. Consider the happy circle defined by the equation $x^2+y^2 = 1$. We want to stretch this into the evil ellipse of death (EEOD), $x^2 + 5y^2 = 1$.

(a) Graph both equations.

(b) For each point below in the circle, find the corresponding point in the EEOD.

Mary, Mary (why you buggin'), quite contrary, how does your circle grow?

$$M \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad R \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \text{and } Y \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(c) Determine a matrix that will transform the circle into the EEOD.

(d) Use scaling to find the area of the EEOD.

(e) Find a cool formula for area of the ellipse with major axis a and minor axis b.

(f) Prove that this transformation matrix forms a real ellipse. For reals.

(g) Find a matrix that will turn the circle into a shape that has as much depth as the career of Justin Bieber.

(h) Find a matrix that will turn the circle into a shape that has as much depth as Justin Bieber's hair. Wait, never mind, this problem is impossible.

4. What transformation matrix takes the parallelogram

$$F \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \quad O \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad U \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \text{ and } \quad R \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

back to a square of side 1 with lower left corner O?

13. (a) Plot the following points.

$$M\begin{bmatrix}1\\6\end{bmatrix},\quad A\begin{bmatrix}0\\0\end{bmatrix},\quad R\begin{bmatrix}7\\2\end{bmatrix},\quad \text{and } Y\begin{bmatrix}8\\8\end{bmatrix}$$

(b) Multiply each point in $MARY$ by the matrix

$$ROSE = \begin{bmatrix}0 & \frac{1}{2}\\ -3 & \frac{7}{2}\end{bmatrix}$$

(c) Plot $ROSE \cdot MARY$

(d) Find all scaled vectors of matrix $ROSE$

(e) How much larger or smaller is the area of $ROSE \cdot MARY$ compared to the area of $MARY$? Try to get this without finding either area.

(f) What will any polygon look like as you multiply it by $ROSE$ repeatedly? Will all the points approach something?

(g) What will happen to the area of polygons repeatedly transformed by $ROSE$?

There's something about Mary. (This one's about the far-away Mary.)

Is this a paradox? Or is it what was stolen from Chance's place?

Double Stuf

7. (s) Let $J(n)$ be the number of ways to tile a 2-by-n rectangle with one-unit-square "mahnaminoes". Find $J(1)$, $J(2)$, and $J(3)$.

(e) Find a closed rule for $J(n)$. Don't think too hard about this one.

(v) Let $M(n)$ be the number of ways to tile a 2-by-n rectangle using any combination of mahnaminoes and dominoes. (Remember, all different orientations are considered different solutions.) Find $M(1)$, $M(2)$, $M(3)$, and $M(4)$.

(2.718...) Find a *three-term* recursive rule for $M(n)$.

(n) Find a closed rule for $M(n)$ if you feel like wading through a giant pit of despair.

There seems to be an echo in here. A *won*-derful problem if you will.

WARNING: This problem is very difficult! There *is* a three-term recursive rule, and a rather nice one.

4. Find a closed rule for

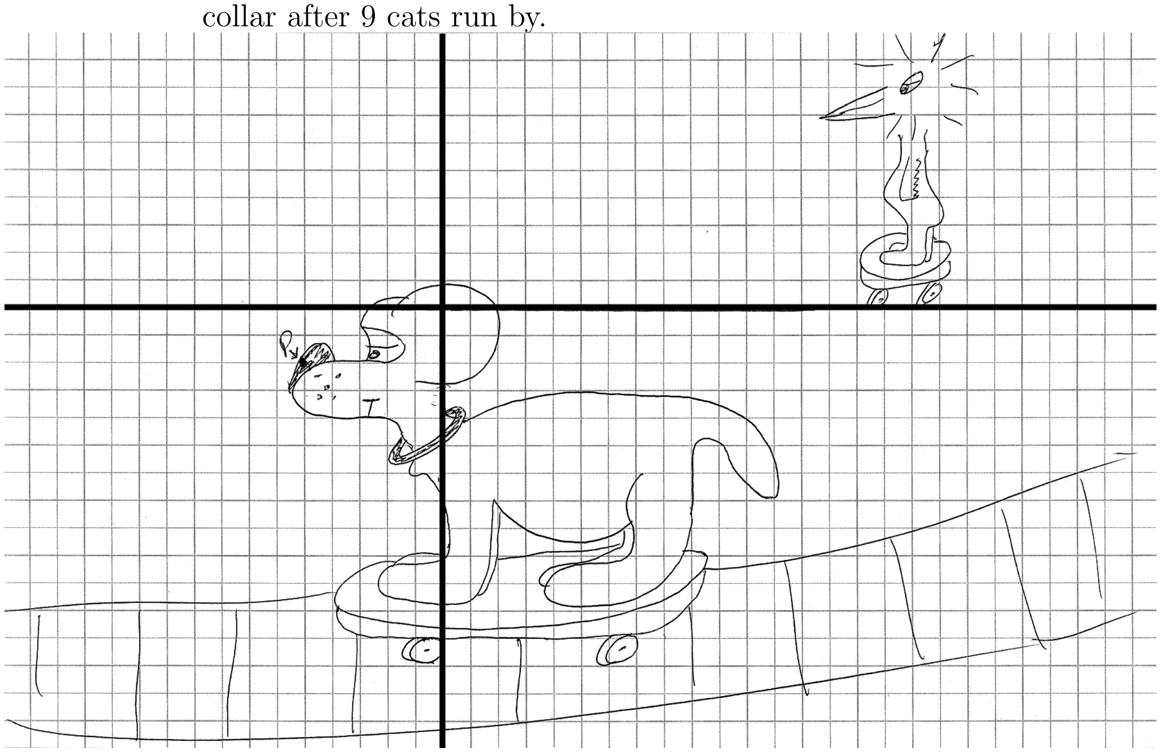
$$\text{Four}(n) = \begin{cases} -4, & n = 0\\ 4, & n = 1\\ 4\cdot\text{Four}(n-1) - 4\cdot\text{Four}(n-2), & n > 1\end{cases}$$

2. (See the provided image below.) Every time a cat runs by, the dog's nose, point $P(-5,-2)$, doubles its distance from the origin while staying along the line $y = \frac{2}{5}x$. At the same time, the bird's wheels stay on the x-axis but reflect to the opposite side of the origin, staying the same distance away.

(a) Find a matrix M that accomplishes this transformation.

(b) Determine the approximate endpoints of the dog's collar after 9 cats run by.

Wait, is someone flipping the bird?



3. (b) Find a matrix D that transforms any point across the line $y = x$.

(o) Find a matrix A, *different from the last one*, that transforms $(5,2)$ to $(2,5)$.

(w) Find a matrix R that transforms any point across the line $y = 3x$.

(e) Find a matrix Y that transforms $(0,0)$ to $(0,1)$.

(n) Find a matrix L that transforms $(0,1)$ to $(0,2)$ *and* $(1,0)$ to $(1,1)$ *and* $(1,1)$ to $(1,2)$.

D.A.R.Y.L. Classic movie! Not a ripoff of WarGames in any way, shape, or form, nope!

11. Start with $KITE$ made from the points $K(-4,-4)$, $I(0,2)$, $T(2,2)$, $E(2,0)$. Careful!

(a) Determine the area of $KITE$.

(b) Determine the area of $KITE$ through careful shearing.

Isn't flying a kite shear fun? If you don't like this joke, go fly a kite!

8. Determine the exact value of

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}$$

12. If you transform a parallelogram through a matrix, will it always remain a parallelogram? Almost always? Are there other shapes that stay the same under transformations?

10ahh. $\begin{bmatrix} 5 & 1 \\ 3 & 3 \end{bmatrix}$

Stare at it. Keep staring at it. That's right. Keep staring. (Stare ∞.)

Yeah, we have no idea what's going on here either. Maybe if we stare long enough, we'll see a 3-D sailboat.

Nim Stuff

12. In one type of *Nim* game, there are two piles of sticks. On a turn, you may take as many as you want from one pile. The goal is to take the last stick. Determine a strategy that wins the game if there are 10 sticks in one pile and 6 in the other. Generalize.

The first game is like moving a rook toward the corner of a large chessboard. Why? What piece is represented in the second game? The second game has a tough but beautiful generalization, by the way.

12. The same game is played again, with one new rule: you may also take *an equal number* of sticks from both piles. The goal is still to take the last stick. Determine a strategy that wins the game if there are 10 sticks in one pile and 6 in the other. Generalize.

12. Celebrate and have a great summer. Thanks for making this course fun, and see you soon.