Lecture 1

Lectures 1 - 7 are based on *The Shape of Space*, by Jeff Weeks

1. Creating universes by gluing

We would like to understand the universe in which we live, as well as other possible universes. People often assume that our universe is $\mathbb{R}^3$ (i.e. an infinite 3-dimensional space that has Euclidean geometry). The actual shape and geometry of our universe is not known. However, there are many other possibilities for our space. The problem is that it is hard to imagine these other possibilities from within our space. To get some perspective on how to do this we consider the analogous problem for 2-dimensional creatures living in a 2-dimensional universe.

There is a well-known book about 2-dimensional space written in 1884, entitled *Flatland*. It was written by Edwin A. Abbott as a commentary on Victorian society. You can download a free copy at http://www.geom.uiuc.edu/ banchoff/Flatland/. It is certainly worth reading, but you have to ignore the Victorian sexism. If you have seen the movie Flatland, you can critique the geometric correctness of various aspects of the movie. For example, the character's faces flip from side to side. Also, when the 2-d creatures are lifted out of Flatland they seem to have 3-dimensional vision.

We will be talking a lot about the main character of *Flatland*, whose name is A. Square. Here is a picture of him:

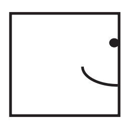


Figure 1. A. Square

A. Square lives in a 2-dimensional space. He and his space have no thickness, like a shadow, or a projected image. In the book *Flatland*, the universe is a plane. However, there are other possibilities.

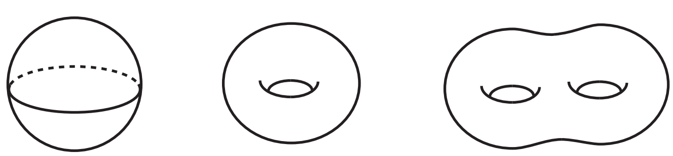


Figure 2. Some possible 2-dimensional universes

For example, the universe could be a sphere, a torus (that is the surface of a doughnut), a 2-holed torus, a 3-holed torus, etc (See Figure 2). These possible universes all have finite area. There are also other 2-dimensional universes which have infinite area. Figure 3 illustrates some examples.

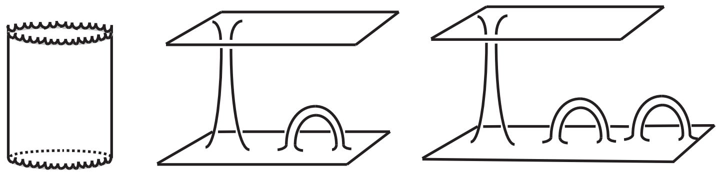


Figure 3. Some possible infinite universes

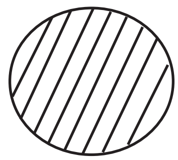


Figure 4. Could a disk be a universe?

Maybe we should consider a disk as a possible universe (see Figure 4). The problem is that a disk has a boundary. What would it be like to live in a universe with a boundary? If you hit the boundary you could go no further. This doesn't seem reasonable. So we won't allow a universe to have boundary. All of the possible shapes for a finite area 2-dimensional universe are known. In fact, there is an infinite list of all finite area 2-dimensional universes. But there is no such list of all finite volume 3-dimensional universes. While many possibilities are known, there could be more that have yet to be thought of. We are going to learn about some of the possible 3-dimensional universes.

It is easy for us to visualize different types of 2-dimensional universes, but hard for 2-dimensional creatures to visualize any universe other than $\mathbb{R}^2$. For example, 2-dimensional creatures can't imagine a torus curving up into the third dimension. We need to give them a way to understand the possibility that a torus is their universe. We shall use the idea of "gluings" and "instant transport" to help them.

Consider a 2-dimensional universe consisting of a giant square where opposite sides of the square are glued together. As 3-dimensional people, we can physically glue opposing sides of a square together (if the square is made of something flexible). We put arrows on the sides to indicate how they are attached. The head of the single arrow is glued to the head of the single arrow, and the head of the double arrow is glued to the head of the double arrow (see Figure 5).

When we glue up the sides of a square we see that we have a torus. The 2-dimensional creatures can't actually glue the sides of the square. So we tell them to imagine that the sides are "abstractly glued" together. This means the sides are not physically glued together, but are glued together in our imagination. Thus if A. Square goes through the top edge he is instantly

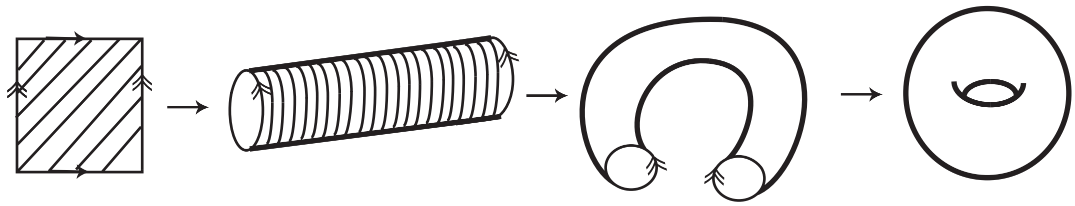


FIGURE 5. A square with opposite sides glued together is a torus.

transported to the bottom edge, like in a video game. Similarly with the left and right sides.

Even though we can physically glue the sides of the square, in order to do so we need the square to be flexible. We can only glue together one pair of opposite sides of a paper square, not both. So we can't make a torus out of paper. We don't want a space to depend on what it is made out of. For this reason, it actually is more convenient for us to think of the torus the way the 2-dimensional creatures do, as an abstractly glued square. This conception of the torus is called a *flat torus*.

We can create an analogous 3-dimensional space by abstractly gluing opposite walls of a cube. For example, consider a room with opposite walls glued together and the floor and the ceiling glued together. We don't want to imagine that the room is flexible and lives in a 4-dimensional space where opposite walls curve around to come together. The concept of this glued up space exists independent of whether a fourth dimension exists or not. This 3-dimensional space is called a 3-torus. If this room were a 3-torus I could walk out the left wall and come back in the right wall. A larger version of a 3-torus could, in fact, be the shape of our universe. If it were our universe, in theory, we could shoot a rocket out in one direction and it would eventually come from the opposite direction (though it might take millions of years to do so). Suppose we had a really powerful telescope what would we see?

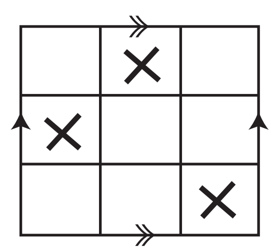


FIGURE 6. 3-in-a-row in torus tic-tac-toe

One way to check your understanding of the flat torus is to play games on a flat torus. For example, we can play torus tic-tac-toe on a flat torus. Figure 6 gives an example of 3-in-a-row in torus tic-tac-toe. Everyone take a minute to play a game of torus tic-tac-toe against your neighbor. If you want to play against a computer, as well as play other games on a flat torus, go to the website:

http://www.geometrygames.org/TorusGames/html/OldTicTacToe.html

Homework

(1) In torus tic-tac-toe, how many inequivalent first moves are there? How many inequivalent second moves are there? Two moves are *equivalent* if the strategy for the rest of the game is analogous. This does not just mean that the NUMBER of ways to win is the same.

(2) Is either player guaranteed to win torus tic-tac-toe, if both players use optimal strategies? Justify your conclusion.

(3) Is it possible to have a tie if both players wanted to? Justify your conclusion.

(4) If you play torus chess and you begin with the usual starting positions will one player necessarily win? Design an opening position that does not lead to an instant checkmate and doesn't automatically lead to a stalemate.

(5) Suppose A. Square lives alone in a small flat torus and each side of A. Square is colored a different color. Imagine that he can see in all directions at once, and has depth perception. He will see infinitely many copies of himself, equally spaced out. Draw a picture that A. Square might draw of how he imagines his space. Such a picture is called an *extended picture* of A. Square's space. Explain your drawing.

(6) Suppose A. Cube lives alone in a small 3-torus and each side of A. Cube is colored a different color. Imagine that he can see in all directions at once, and has depth perception. Draw an extended picture of A. Cube's space. Explain your drawing.

(7) Suppose A. Square lives alone in a small sphere. What direction would he look if he wanted to see the top of his head?

(8) Suppose A. Square lives in a small sphere. He has a friend who walks away from him. Does his image of his friend get bigger or smaller as he walks away? At what point is his image of his friend the smallest?

    Suppose A. Square lives at the north pole in a small sphere. He has a friend who lives at the south pole. What does his friend look like to him?

(9) Suppose A. Square lives alone in a small sphere and each side of A. Square is colored a different color. Imagine that he can see in all directions at once, and has depth perception. Draw an extended picture that A. Square might draw representing his space. How many copies of himself would he see? In particular, if he sees copies of himself, draw all of the copies of himself that he imagines live in his space with him.

(10) Suppose that A. Square is told that his universe is either a flat torus, a sphere, or a plane. What information would help him to determine which it really is?

(11) How could A. Square detect the difference between a plane, an infinite cylinder, and an infinite cone? Is an infinite cone or an infinite cylinder a reasonable 2-dimensional universe for A. Square to live in? Why or why not?

(12) How do we create an infinite cylinder and an infinite cone by gluing up a planar surface? What might be the 3-dimensional analogue of an infinite cylinder? What about an infinite 3-dimensional cone?

(13) Suppose we were told that our universe was either a 3-torus, or $\mathbb{R}^3$. What information would we look for to determine which it really was?

Lecture 2

1. Visualizing higher dimensions

We would like to use our understanding of dimensions 1, 2, and 3, to try to visualize a fourth dimension. People often like to think of the fourth dimension as time. If I want to meet someone I have to specify not only three spatial coordinates (e.g., on the second floor of the building on Broadway and 95th street), but also a time coordinate (e.g. at 9:00 AM on June 29th). In fact, no matter how many spatial dimensions you live in you can always add a time dimension. So time isn't really *the* fourth dimension, it is an $(n+1)^{th}$ dimension for anyone who lives in n dimensions.

However, considering time as a dimension does not help us visualize 4-dimensional space. Also, mathematicians like symmetry so we prefer all dimensions to be equivalent. There is no particular dimension which is the first dimension, and another which is the second dimension, and so on. Rather, dimensions 1, 2, and 3 are all equivalent. So dimension 4 should be equivalent to the others as well.

To understand a fourth spatial dimension, let's see how Flatlanders could think about a third spatial dimension. In order to prove to A. Square that a third dimension exists we could play some tricks on him. Suppose A. Square has a flower in his garden. We could remove the flower from his 2-dimensional space just by pulling it up into the third dimension. It would seem to disappear into thin air. Similarly, a 4-dimensional creature could pull my desk out of my 3-dimensional space. It would seem to me that the desk had disappeared into thin air.

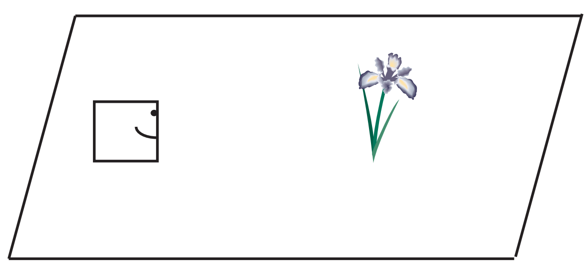


Figure 1. We could steal A. Square's flower and it would just disappear.

If we were really evil we could steal A. Square's heart. But if we did that, a 4-dimensional creature could steal my heart. So let's not do that.

Suppose we grab A. Square, turn him over, and put him back in Flatland. Could he tell what happened? Could the other Flatlanders notice? How? What would happen if a 4-dimensional creature turned me over? Would I know what happened? Would you notice by looking at me?

Figure 2. We could steal A. Square's heart.

2. 4-dimensional geometric figures

One way to visualize higher dimensions is to think about specific geometric shapes. A *hypercube* is a 4-dimensional cube. To visualize a hypercube, we inductively construct a cube in each dimension. (What does it mean to construct something "inductively"?)

We start with a 0-dimensional cube, which is just a single point. Then we construct a 1-dimensional cube by dragging a 0-dimensional cube along in a straight path.

Figure 3. A 1-dimensional cube.

We construct a 2-dimensional cube by dragging a 1-dimensional cube along in a straight path in a direction which is perpendicular to the 1-dimensional cube.

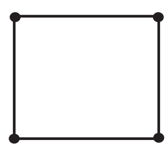


Figure 4. A 2-dimensional cube.

We construct a 3-dimensional cube by dragging a 2-dimensional cube along in a straight path in a direction which is perpendicular to the 2-dimensional cube.

The Flatlanders don't understand Figure 5, because they have no 3-dimensional perspective. So we can draw a 3-dimensional cube for them as in Figure 6, and tell them that the inner square is "further away in the third dimension". They don't really know what further away in the third dimension means, but they can accept this picture.

We construct a 4-dimensional cube by dragging a 3-dimensional cube along in a straight path in a direction which is perpendicular to the 3-dimensional cube. Since we 3-dimensional people have no 4-dimensional perspective, we have to draw a 4-dimensional cube as in Figure 7, where the inner cube is 'further away in the fourth dimension".

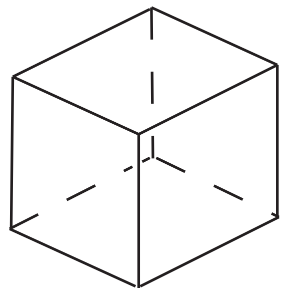


FIGURE 5. A 3-dimensional cube.

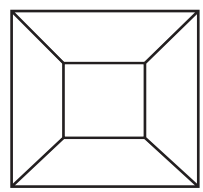


FIGURE 6. A Flatland drawing of a 3-dimensional cube.

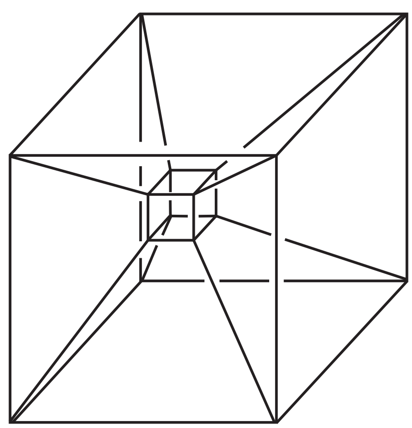


FIGURE 7. A 3-dimensional drawing of a 4-dimensional cube.

We will use a different inductive construction to visualize a 4-dimensional tetrahedron. We start with a 0-dimensional tetrahedron, which is just a point. We add a new point and connect it with a straight line to the previous point to get a 1-dimensional tetrahedron.

FIGURE 8. A 1-dimensional tetrahedron is the same as a 1-dimensional cube

Now we add a new point which is equidistant from the previous two points. We connect this new point to the previous points to get a 2-dimensional tetrahedron (see Figure 9).

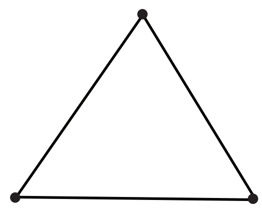


FIGURE 9. A 2-dimensional tetrahedron

Now we add a new point which is equidistant from the previous three points. In order to do this, we put our new point behind the plane of the paper. We connect this new point to the three previous points to get a 3-dimensional tetrahedron (see Figure 10). This is what we usually think of as a tetrahedron.

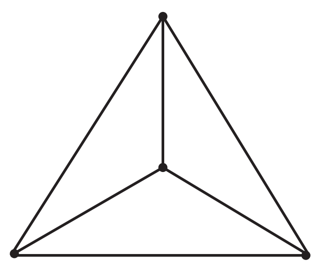


FIGURE 10. A 3-dimensional tetrahedron

The Flatlanders see Figure 10 as an equilateral triangle with a vertex in the center connecting to the other three vertices. We explain to them that even though the new segments look shorter, the central point is actually in the third dimension so that all the segments have the same length.

Now we add a new point which is equidistant from the previous four points. We connect this new point to the previous four points to get a 4-dimensional tetrahedron. Since we have no 4-dimensional perspective, we have to draw it by putting the new point in the center of the 3-dimensional tetrahedron (as in Figure 11). However, we should realize that the central point is actually in the fourth dimension so that all the segments have the same length, even though the new segments look shorter than the previous segments. We can create other higher dimensional figures using similar inductive constructions.

Another way to explain a cube to Flatlanders is to unfold a 3-dimensional box into two dimensions. When we unfold it we label the edges so that we will know how to glue it up again. So edge 1 is glued to the other edge 1, edge 2 is glued to the other edge 2, and so on. This is just like when you

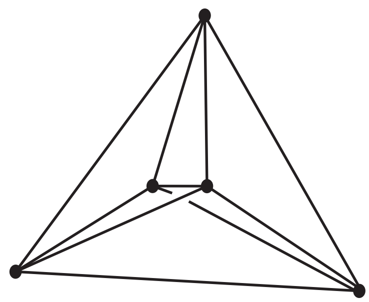


FIGURE 11. A 3-dimensional drawing of a 4-dimensional tetrahedron

are moving and you take apart your wall unit and label the pieces so that you will know how to put it back together again.

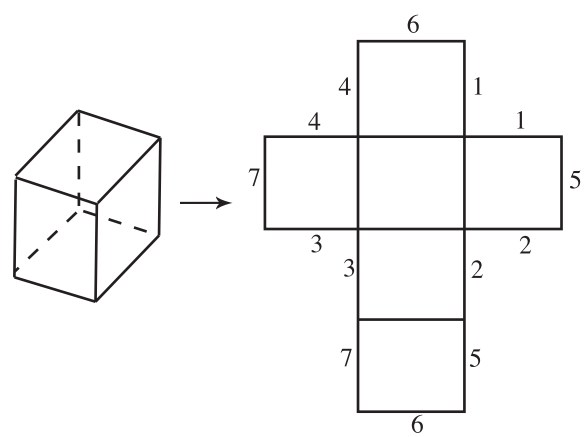


FIGURE 12. Unfolding a cube

In the homework you consider how to glue up an unfolded hypercube. You may also want to read the short story "And he built a crooked house" by Robert Heinlein, which is about people living in an unfolded hypercube, which folds back up while they are in it. You can download a free copy at:

http://arch.ced.berkeley.edu/courses/arch239a/Resources/crooked%20house%20story.htm

Homework

(1) Consider the unfolded hypercube below. Fill in the missing numbers to show how the sides are glued together to produce a hypercube.
(2) What does A. Square see if a 3-dimensional person slowly puts his/her hand into Flatland? What might we see if a 4-dimensional person slowly stuck his/her hand in our space?

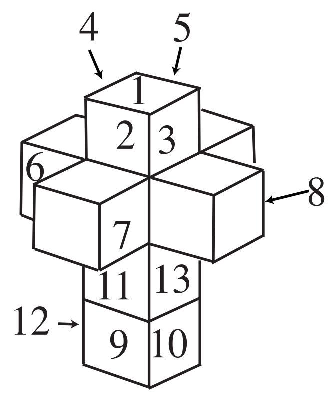


FIGURE 13. An unfolded hypercube

(3) Suppose a cube passes through Flatland. What are all the possible 2-dimensional shapes that a Flatlander might see?

(4) Design a 2-dimensional room with a lock on the door so that the people inside can lock themselves in. Note that there is no gravity in Flatland. So you cannot use the weight of the door as a lock.

(5) Design a digestive system for A. Square so that he would not fall apart. For the sake of health, A. Square's digestion must have a separate entrance for food and exit for waste, both of which might be open at the same time in case of illness. Also you cannot assume that food is simply absorbed through a membrane.

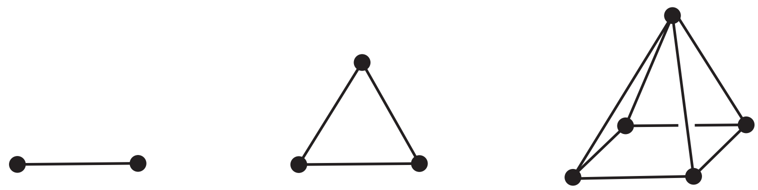
(6) If we lift A. Square above Flatland and hold him perpendicular to it, what will he see?

(7) If everything in Flatland had the same small amount of 3-dimensional thickness, including people's eyes, could A. Square detect it? If everything in our universe had the same small 4-dimensional thickness could we detect it?

(8) What would we see if another 3-dimensional space intersected ours perpendicularly? How would we see the 3-dimensional creatures of that space?

(9) How many vertices, edges, faces, and solids does a 4-dimensional cube have and why? What about a 4-dimensional tetrahedron? Find recursive formulae for the number of vertices, edges, faces, and solids of an n-cube and an n-tetrahedron. Justify your conclusions.

(10) An *n-dimensional pyramid* consists of an (n-1)-dimensional cube with all vertices attached to a single additional point in the nth dimension. See the figure below. Determine how many vertices, edges, faces, and 3-dimensional solids a 4-dimensional pyramid and 5-dimensional pyramid have. Justify your conclusions.



1-dimensional 2 -dimensional 3-dimensional

Lecture 3

1. Vocabulary of Geometry and Topology

Now we return to our study of 2-dimensional and 3-dimensional spaces. In order to be able to describe the differences between different spaces, we need to introduce some terminology. We will give intuitive rather than rigorous definitions of each of the terms involved.

Definition 1. *A space is said to be an* n**-manifold** *if, in any small enough region, it looks like* $\mathbb{R}^n$.

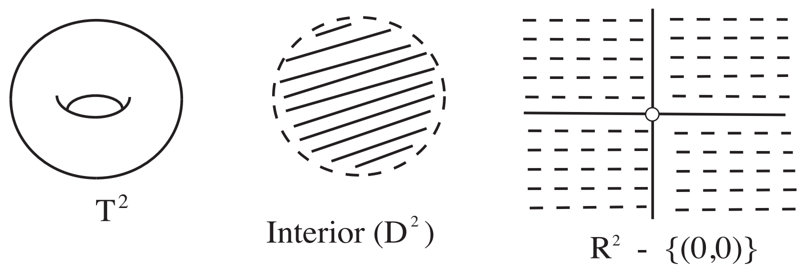


Figure 1. Some 2-manifolds

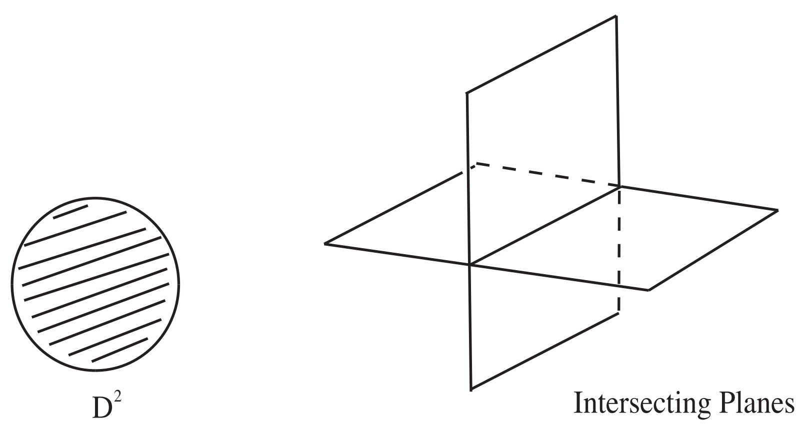


Figure 2. Some non-manifolds

What do we mean by "small enough"? Why aren't the spaces in Figure 2 manifolds? We want to make reasonable assumptions about what sorts of objects can be universes. So we make the following rules:

Rule 1. No universe has a boundary

Rule 2. The universe of an n-dimensional creatures is an n-manifold.

In fact, Rule 1 follows from Rule 2. How? From Rule 2 we know that Flatland is a 2-manifold, and our universe is a 3-manifold. We will add more rules later. First we need to introduce some more terminology.

Definition 2. *The* **topology** *of a manifold is the set of properties of it which are unchanged by deforming it or cutting it apart and regluing the same parts together.*

This definition will become clearer if we consider some examples. The following two spaces have the same topology.

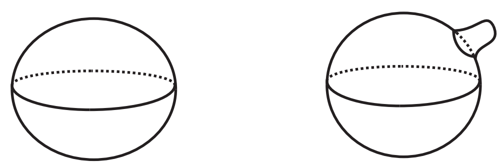


FIGURE 3. These spaces have the same topology

A solid doughnut has a different topology than a ball, but the same topology as a coffee cup. Which is why people say that topologists can't tell the difference between a doughnut and a coffee cup.

FIGURE 4. A doughnut has the same topology as a coffee cup

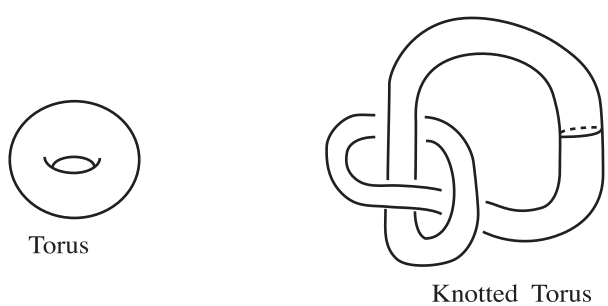


FIGURE 5. A torus has the same topology as a knotted torus

An ordinary torus and a knotted torus have the same topology, since we can cut the torus and reglue the same points together so that it now has a knot in it (see Figure 5). These two spaces look different to us, because they are situated differently in our space. But their 2-dimensional topology is actually the same.

An example of a topological property of a sphere or $\mathbb{R}^2$ is that any loop of string they can be gathered up into a ball. In a torus a loop of string that goes around the hole cannot be gathered up into a ball. This shows that the torus is topologically different from the sphere.

FIGURE 6. A loop on a sphere can always be gathered into a ball. A loop around the hole of a torus cannot be gathered into a ball.

Another topological property is how many circles you need to cut along so that a surface can be opened up and deformed into a plane. For example, for a torus you need to make two such cuts, while for a 2-holed torus you would need more than two such cuts.

Definition 3. *The* **geometry** *of a space is the set of properties of the space which can be measured. This includes distances, areas, volumes, angles, curvature, etc..*

The spaces in Figure 7 all have different geometry. Why?

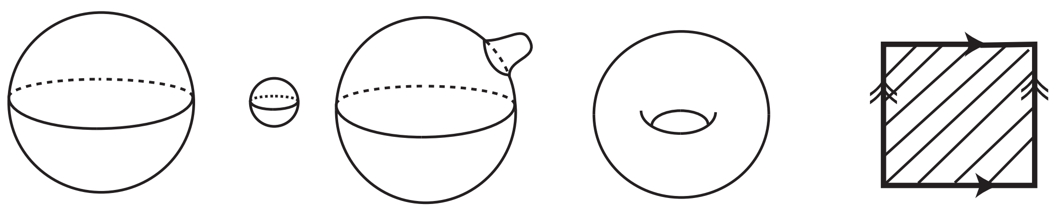


FIGURE 7. These spaces all have different geometry

Definition 4. *Let P be a path in a space. We say that P is a* **geodesic** *if for any two points on P which are close enough together on P, the shortest path between the two points lies on P.*

A geodesic is the path made by a ray of light or a string puled taut on the surface. Figure 8 illustrates a geodesic P on a cone. The shortest path between the points x and y is not on the P because these two points are not close enough together on P (though they are close together on the cone). Distances do not change when a cone is cut open into a flat surface. So a

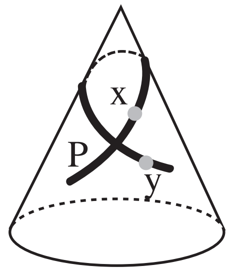


FIGURE 8. The path P is a geodesic on this cone. But the shortest distance between x and y is not on P

geodesic on a cone is a straight line when the cone is cut open. In particular, a circle which is parallel to the base of the cone is not a geodesic.

Geodesics will play the role of straight lines in our spaces. A geodesic may contain more than one path between two points. On a sphere the geodesics are great circles. There are infinitely many geodesics between the north and south pole or any other pair of opposite points on a sphere.

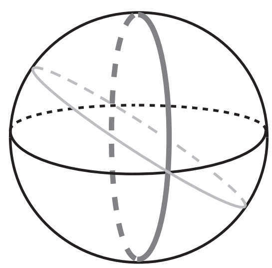


FIGURE 9. Some geodesics on a sphere

Consider the space $\mathbb{R}^2 - \{(0,0)\}$. There is no geodesic between the points p and q in see Figure 10. What would it be like to live in a universe with a point missing?

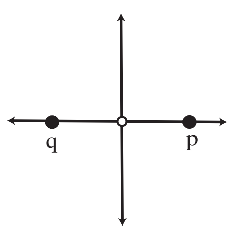


FIGURE 10. The points p and q in $\mathbb{R}^2 - \{(0,0)\}$ have no geodesic between them

Definition 5. *The* **intrinsic** *properties of a space are those properties which can be detected by people who live in the space. The* **extrinsic** *properties of* *a space are those that have to do with how the space is situated in higher dimensional space. Extrinsic properties cannot be detected by people who live in the space.*

Spaces which are the same extrinsically must be the same intrinsically. This is because if you can tell they are the same from the outside, then they must actually be the same. But the converse is not true. For example, an ordinary torus and a knotted torus have the same intrinsic topology but different extrinsic topology (see Figure 5). Extrinsically, we can tell the difference between the two tori because one cannot be deformed to the other in $\mathbb{R}^3$. In general, if one surface can be deformed to another in $\mathbb{R}^3$ then they have the same extrinsic topology. If a surface can be cut apart and re-glued so that the same points are glued together again then the surfaces have the same intrinsic topology but not necessarily the same extrinsic topology.

Intrinsic differences can be discussed over the phone by creatures living in the spaces, extrinsic differences can't be. For example, the difference between a sphere and a torus is intrinsic. But how can it be detected by creatures living in the space and talking to one another on the phone? The topological properties of crumbling up a loop of string and of the number of cuts needed to make the space planar are both intrinsic properties, because they do not have to do with how the space is situated in a higher dimension.

A right handed helix and a left-handed helix have the same intrinsic and extrinsic topology, since one can be deformed to the other. The intrinsic geometry is also the same, since one is the mirror image of the other and hence all measurements within the space will be identical (we assume that all creatures in all universes have the same measuring devices so that they can compare lengths, areas, angles, etc. when they are talking about their spaces on the phone). If creatures in one helix talk on the phone to creatures in the other helix, they cannot tell their spaces apart. On the other hand the helices have different extrinsic geometry because we can see from the outside that they are curving in different directions (see Figure 11). In general, two spaces have the same extrinsic geometry if and only if one can be rigidly superimposed on the other.

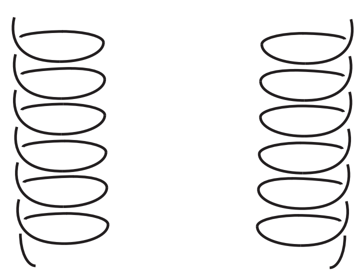


FIGURE 11. These helices have the same intrinsic geometry but different extrinsic geometry.

One way to understand the intrinsic geometry of a space is in terms of angles. In particular, we consider whether the sum of the angles of triangles adds up to more, less, or equal to 180°. We can also consider whether there are 360° around every point. We begin by thinking about what we mean by a triangle. A triangle is a region with three sides, each made of a geodesic segment. For example, consider a triangle on a sphere. Then the sides of the triangle are segments of great circles (ie., pieces of equators).

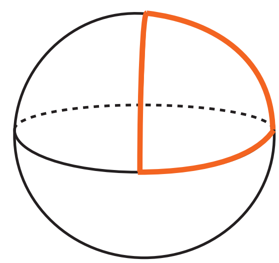


FIGURE 12. Here is a triangle whose angle sum is 270°.

The sum of the angles of the triangle in Figure 12 is 270°. In fact, there is no triangle on a sphere with angle sum equal to exactly 180°. The angle sum is always greater than 180°; and the bigger the triangle on the sphere, the greater the angle sum (see Figure 13).

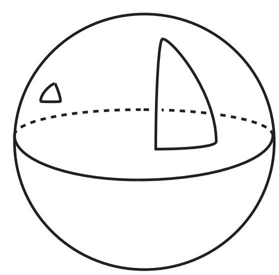


FIGURE 13. The bigger the triangle, the greater the angle sum.

On the other hand, there are also surfaces where the sum of the angles of any triangle is less than 180°. For example, Figure 14 illustrates a picture of an infinite saddle surface, where the angle sum of any triangle is less than 180°, and the bigger the triangle the smaller the angle sum.

In both of the above examples very tiny triangles look almost flat, and have angle sums which are almost equal to 180°. So a small creature living in one of these surfaces might think that all triangles have angle sums of 180°. Also, a plane and a saddle surface have the same intrinsic and extrinsic topology because one can be deformed to the other. However, because they have different types of triangles, their intrinsic geometry is different, and because one cannot be superimposed on the other their extrinsic geometry is different.

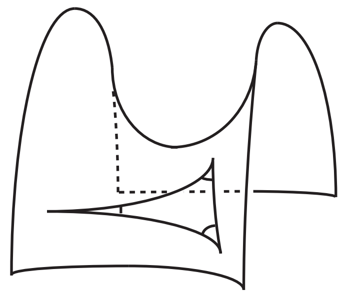


Figure 14. A triangle whose angle sum is less than 180°.

Let's consider the geometry of a torus. On a flat torus, the geodesics are ordinary line segments. So every triangle has angle sum 180°, even if the triangle cuts across the gluing line. If we consider the curved torus in $\mathbb{R}^3$, the geodesics are no longer straight segments. The curved torus has some triangles whose angles add up to more than 180° and other triangles whose angles add up to less then 180° (see Figure 15), and if you look carefully you could even find one that adds up to exactly 180°. So there is no general statement that we can make about angle sums of triangles in the curved torus. This is another reason why a flat torus is more desirable than a curved torus.

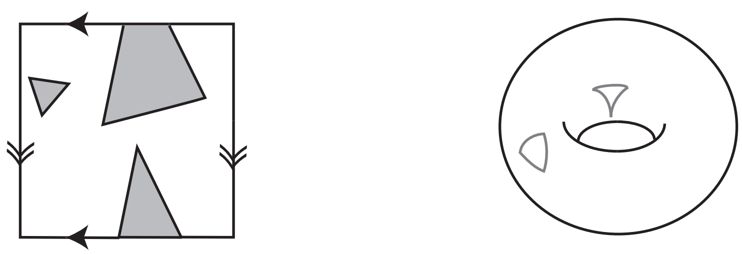


Figure 15. Triangles on a flat and a curved torus.

Definition 6. *A property of a point in a space is said to be* **local** *if it can be detected in ANY sufficiently small region containing that point. A property that is not local is said to be* **global**.

For example, the property of being an n-manifold is a local property at every point, since in any small enough region an n-manifold looks like $\mathbb{R}^n$. Thus the local topology of all 2-manifolds is the same. But the global topology of a sphere and a torus are different. A big sphere and a small sphere have different intrinsic global geometry, because the difference in total area cannot be detected locally.

Consider the two infinite surfaces in Figure 16. Since they are both 2-manifolds they have the same local intrinsic topology. Also, since the 2-manifolds are sitting in a 3-manifold they have the same extrinsic local topology, and since one can be deformed to the other they have the same global extrinsic (and hence intrinsic) topology. However, they have different local and global intrinsic geometry. Why? What about their extrinsic local and global geometry?

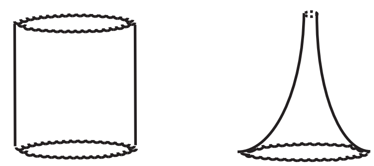


FIGURE 16. These spaces have the same topology but different geometry.

Observe that in order to understand the extrinsic properties of a surface we have to be outside of the surface. Hence nobody can ever study the extrinsic properties of their own space. Furthermore, if the space is not embedded in a larger space, then it has no extrinsic topology or geometry because there is nothing outside of it space. For example, a flat torus and the flat 3-torus (obtained by gluing opposite walls of a room) have no extrinsic properties, because they are abstractly glued-up shapes. In studying the possible shapes of our universe we prefer to take the *intrinsic viewpoint*. That is, we assume that all spaces exist in and of themselves, without assuming that they are embedded in some higher dimensional space.

We shall be most interested in the intrinsic global topology and the intrinsic local geometry of spaces. Why?

Definition 7. *An n-manifold is said to be* **homogeneous** *if its local geometry is the same at all points.*

If a space is homogeneous, then the sum of the angles of a triangle of a given size is the same wherever the triangle is, and the sum of the angles around a point is always $360°$. For example, the flat torus is homogeneous. On the other hand, the curved torus is not homogeneous since the sum of the angles of triangles of the same size differs depending on where the triangle is on the surface (see Figure 17). What about an infinite cone?

FIGURE 17. A flat torus is homogeneous, while a curved torus is non-homogeneous.

It is hard to study the geometry of non-homogeneous spaces because you can't make any general statements about the space. Also, it does not seem reasonable for the geometry of a universe to change from one place to another. Thus we add the following to our list of rules about universes.

Rule 3. All universes are homogeneous.

Definition 8. *An n-manifold is* **flat** *if it has the same local geometry as* $\mathbb{R}^n$*. Otherwise it is* **curved***.*

Note that being flat or curved is an intrinsic property. This is different than saying that a surface (like a torus) "curves" in $\mathbb{R}^3$, which is an extrinsic property. Flat geometry is the same as Euclidean geometry, which means that the sum of the angles of triangles is always 180° in any flat manifold. List all the flat manifolds that we know? Is an infinite cylinder flat? It can be made by gluing together an infinite strip.

Homework

(1) Make a table comparing the local, global, intrinsic, and extrinsic topology and geometry of the circular strips illustrated below.

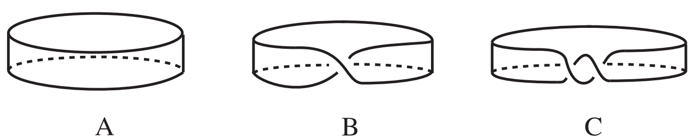


FIGURE 18. Compare these figures.

(2) Using one or more pieces of paper, can you make a surface containing a triangle whose angles add up to more than 180°? Can you make a surface out of paper with a triangle whose angles add up to less than 180°? Can you make a surface out of paper that contains some triangles whose angles add up to 180°, some which add up to less than 180°, and some which add up to more than 180°? Either include your paper constructions with your homework or clearly describe them.

(3) From an intrinsic point of view, compare the local and global, geometry and topology of a disk and a finite cone, both with boundary. How could Flatlanders distinguish these two spaces?

(4) Give examples (different from the ones given in these notes) of pairs of spaces with the following properties.
a) The same local intrinsic topology, but different local intrinsic geometry.
b) The same local intrinsic geometry, but different global intrinsic topology.

c) The same global intrinsic topology, but different global extrinsic topology.
d) The same local intrinsic geometry, but different global intrinsic geometry.
e) The same global intrinsic geometry, but different global extrinsic geometry.

(5) Is the surface of a cube homogeneous? Why or why not?

(6) Compare the local and global geometry and topology of a 3-torus made from a rectangular room and one made from a cubical room.

(7) For each of the surfaces listed below, state one or more intrinsic topological and/or geometric properties that enables you to distinguish that surface from ALL of the other surfaces in the list. State whether each property you refer to is topological or geometric.
- a plane
- a flat torus
- a flat Klein bottle
- a sphere
- the surface bounding a cube

Lecture 4

1. Orientability

We have seen that if a person is turned over in a higher dimension then the person will become his/her mirror image. It is also possible to become your mirror image without even leaving your space. Here is a picture of a Möbius strip. You can see that A. Square gets reversed when he walks completely around the strip.

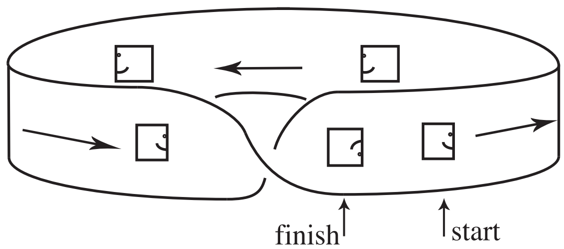


Figure 1. A. Square walks around a Möbius strip.

Definition 1. *Any path that you can walk along and have your orientation reversed is called an* **orientation reversing path**. *A space is said to be* **non-orientable** *if it contains an orientation reversing path. Otherwise it is said to be* **orientable**.

A Möbius strip is non-orientable, however is not a 2-manifold because it has a boundary. We would like to create a similar space without a boundary. First we represent a Möbius strip by gluing as in Figure 2.

Figure 2. A Möbius strip.

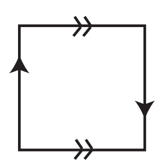


Figure 3. A Klein bottle.

To get rid of the boundary we glue the boundary to itself as in Figure 3. This space is called a *Klein bottle*. Since it contains a Möbius strip, it has an orientation reversing path, and hence it is non-orientable. Similarly, we can create a 3-dimensional Klein bottle by gluing a room with the front wall glued to the back wall and ceiling glued to the floor as in the 3-torus, but with the side walls glued together with a flip so that the back of the left wall is glued to the front of the right wall. What would happen to me if I walked through the left wall?

This is an interesting space to study. But it doesn't seem likely that people can actually take a trip around their universe and come back reversed. So we make another rule about real universes.

Rule 4. All universes are orientable.

We can play Tic-Tac-Toe on a Klein bottle, just as we can on a torus. The easiest way to analyze the moves is to draw an extended diagram with numbers in the squares so that identical numbers represent identical squares. You can use the extended diagram in Figure 4 to find some interesting diagonals on the Klein bottle Tic-Tac-Toe board.

7	8	9	1	2	3	7	8	9
4	5	6	4	5	6	4	5	6
1	2	3	7	8	9	1	2	3
7	8	9	1	2	3	7	8	9
4	5	6	4	5	6	4	5	6
1	2	3	7	8	9	1	2	3
7	8	9	1	2	3	7	8	9
4	5	6	4	5	6	4	5	6
1	2	3	7	8	9	1	2	3

Figure 4. An extended diagram of a Klein bottle tic-tac-toe board.

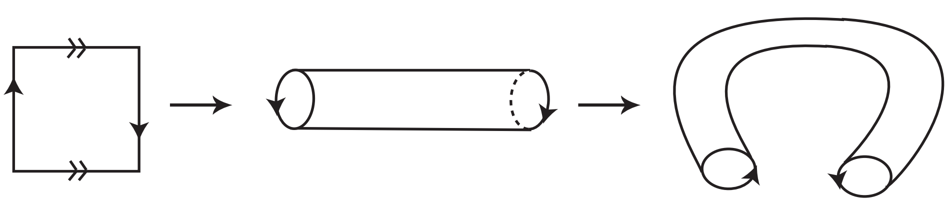


Figure 5. An attempt to glue up a Klein bottle.

What happens if we try to glue up a Klein bottle in $\mathbb{R}^3$? (See Figure 5). In order to match the circles together, we would have to bring one end up into a fourth dimension so that it could approach the other end from the inside. The tube in Figure 6 is not supposed to intersect itself, but this is the only way we can draw it in $\mathbb{R}^3$. However, we prefer to take the intrinsic viewpoint, and just draw the Klein bottle as in Figure 3. In this way, we don't have to worry about what dimensional space the Klein bottle lives in, and we don't have to distort the geometry of the Klein bottle as we bend it around.

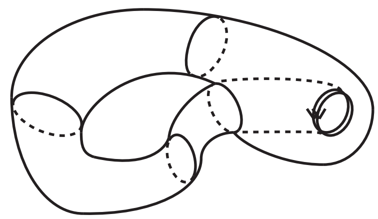


FIGURE 6. A drawing of a Klein bottle.

Note that when we look at a glued up polygon it is not always obvious what surface it represents. However, it is easy to check if the surface is orientable by seeing if it contains a Möbius strip. Consider the glued up hexagon in Figure 7. It is easy to see that this surface contains a Möbius strip and hence is non-orientable.

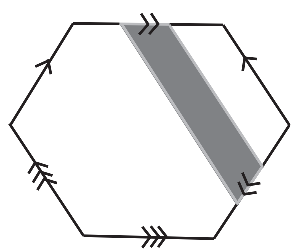


FIGURE 7. A non-orientable surface.

Another important non-orientable surface is the *projective plane*, P^2 or $\mathbb{R}P^2$. We obtain P^2 from a disk by gluing opposite points on the boundary of the disk as indicated in Figure 8. One way to understand th eprojective plane is to cut out a disk to get a Möbius strip (as in Figure 9). Thus the projective plane is a Möbius strip with a disk attached along the boundary. Since P^2 contains a Möbius strip it is non-orientable. Like the Klein bottle, the projective plane can't be embedded in $\mathbb{R}^3$.

Is the projective plane homogeneous? Inside the original disk it has flat geometry, but a triangle which contains points along the boundary of the disk has an angle sum which is greater than 180° (see Figure 10).

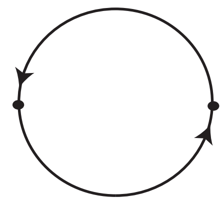


FIGURE 8. A projective plane.

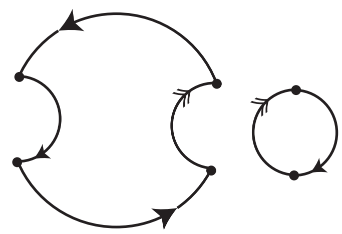


FIGURE 9. A projective plane can be cut up into a Möbius strip and a disk.

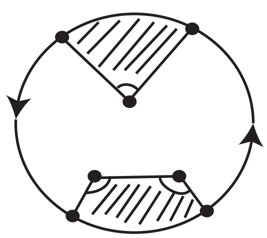


FIGURE 10. This triangle has angle sum greater than 180°.

To avoid this problem we choose to make P^2 out of a glued-up hemisphere rather than a glued up disk (see Figure 11). When we do this, the local geometry of P^2 is the same as the local geometry of a sphere, where the angle sum of a triangle is always more than 180°. This makes the projective plane homogeneous. We prefer this construction of the projective plane just as we prefer the flat torus to the curved torus.

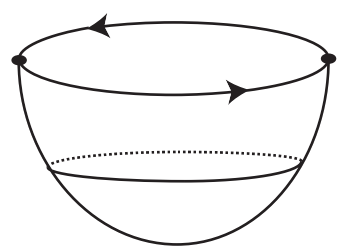


FIGURE 11. A homogeneous projective plane.

Analogously, we can create a 3-dimensional projective space, P^3. Instead of starting with a disk (or a hemisphere which has the same topology as a disk). A disk is the set of all points in $\mathbb{R}^2$ which are a distance less than or equal to one from the origin, and a ball is the set of all points in $\mathbb{R}^3$ which are a distance less than or equal to one from the origin. To obtain P^3, we glue opposite points along the boundary of a ball, as we did along the boundary of the disk in Figure 12).

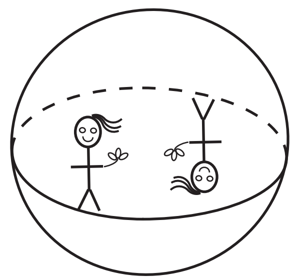


FIGURE 12. To obtain P^3, we glue opposite points on the boundary of a ball.

2. ORIENTABILITY AND 2-SIDEDNESS

We temporarily leave our intrinsic viewpoint to consider the following property of a 2-manifold as it is situated in a 3-manifold.

Definition 2. *A 2-manifold contained in a* 3*-manifold is said to be*1**-sided** *if there is a path on the surface that a* 3*-dimensional creature can walk along to come back on the other side. Otherwise it is said to be* 2**-sided***.*

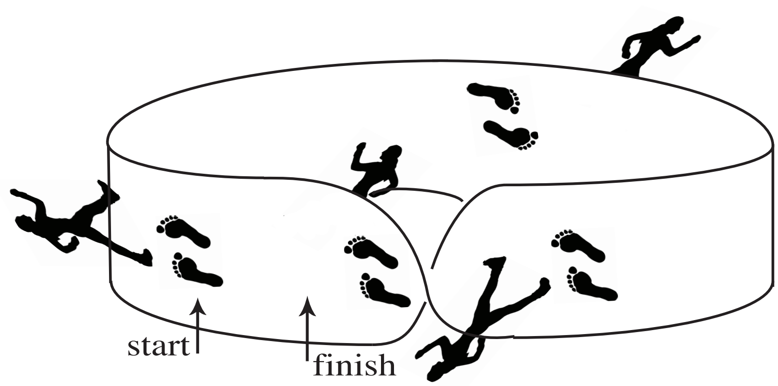


FIGURE 13. I walk around a Möbius strip, and come back on the other side.

Observe that 1-sidedness is an extrinsic property of a surface in a 3-manifold because it has to do with how the surface is embedded. Let's consider a Möbius strip embedded in $\mathbb{R}^3$. Suppose I walk around a Möbius strip. My footprints will return to where they started but I will be on the other side of the Möbius strip (see Figure 13). Thus this embedding of a Möbius strip in $\mathbb{R}^3$ is 1-sided. On the other hand, if I stand on a cylinder and walk around in a circle, I will never come back on the other side. So a cylinder in $\mathbb{R}^3$ is 2-sided.

Now we consider a 3-dimensional Klein bottle K^3 obtained by gluing the top and bottom of a cube with a left-right flip, and otherwise gluing opposite sides straight (see Figure 14). In Figure 14 we consider three intersecting surfaces and determine whether each is orientable and/or 1-sided. This example shows that various combinations of orientable or non-orientable and 1-sided or 2-sided are possible. By contrast, when the 3-manifold containing the surface is orientable, we have the following theorem.

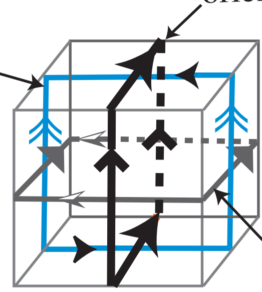


FIGURE 14. orientability is independent of 1-sidedness

Theorem. *Suppose S is a surface in an orientable 3-manifold M. Then S is 1-sided if and only if S is non-orientable.*

Proof. ($\Rightarrow$) Suppose that S is 1-sided in M. A 3-dimensional person (let's call her Erica) walks along a path on S so that she ends up on the other side of S. We consider the foot prints of her left foot. Since M is orientable, we know that Erica will not be reversed. So her left foot remains on her left side throughout the journey. We don't know what the surface or the manifold looks like. However because Erica's journey has taken her from one side of the surface to the other side of the surface, we know that at the start and finish of her trip her left footprints look like they do in Figure 15. However, from the point of view of the Flatlanders who live in the surface, Erica's journey has caused the left footprint (as a 2-dimensional creature) to become reversed. Thus the surface has an orientation reversing path, and hence is non-orientable.

($\Leftarrow$) Now suppose that S is a non-orientable surface in M. Then S contains a path which reverses 2-dimensional creatures. Erica walks along this

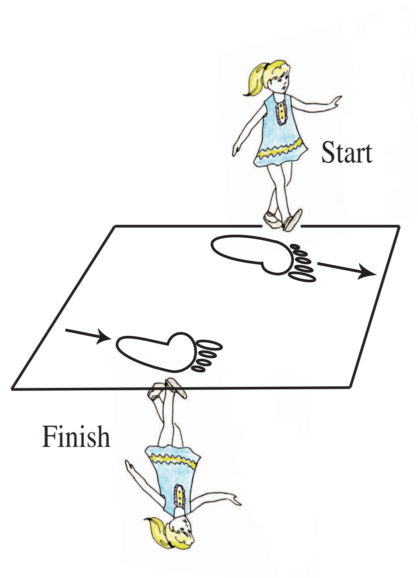


FIGURE 15. Erica comes back on the other side, and her left foot print is still a left foot print.

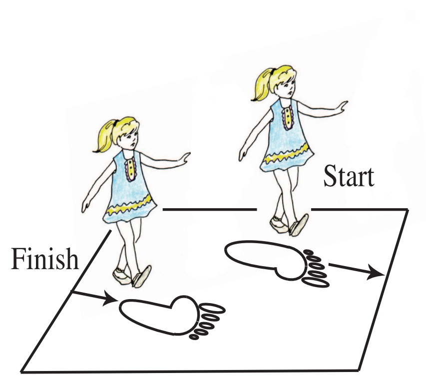


FIGURE 16. Erica stays on the same side of the surface, but her left footprint becomes a right footprint.

path, and her left foot leaves footprints. These footprints are 2-dimensional creatures, and hence will be reversed by their trek along the orientation reversing path. If Erica stayed on the same side of the surface throughout her journey and her left footprint became reversed, it would become a right footprint (see Figure 16). This would mean that Erica was reversed. But this is impossible since the 3-manifold M is orientable, and hence there is no path that reverses an 3-dimensional person. Thus in fact, her walk must have taken Erica to the other side of the surface. Hence the surface is 1-sided. $\square$

Homework

(1) Does a 3-dimensional Klein bottle necessarily contain a 2-dimensional Klein bottle? If so, where? And how many Klein bottles does it contain?

(2) Carefully explain whether or not a 3-dimensional projective space is orientable. Draw a picture to illustrate your argument.

(3) A tube connecting two spaces is sometimes called an *Einstein-Rosen bridge*. What does an Einstein-Rosen bridge look like to Flatlanders?

(4) Suppose two parallel planes are connected by an Einstein-Rosen bridge. How do the squares in one plane appear to the squares in the other plane? If the squares in the two planes are oriented in a parallel way, do the squares in one plane appear to the squares in the other plane to be reversed? Is the space orientable? If two planes are connected by two Einstein-Rosen bridges is the space orientable?

(5) Describe a 3-dimensional Einstein-Rosen bridge. Suppose there is a 3-dimensional Einstein-Rosen bridge attaching one $\mathbb{R}^3$ to a parallel $\mathbb{R}^3$. Is the space orientable?

(6) Suppose A. Square lives on a Möbius strip. How does he see himself? Does he appear to be reversed? Suppose he stands still and watches his friend B. Triangle walk around and around the strip. Just as his friend is passing him after going once around the strip, A. Square looks at all the copies of his friend. Which, if any, of them will appear to be reversed? Draw an extended picture of a Möbius strip containing A. Square and B. Triangle.

(7) Suppose you live in a 3-dimensional Klein bottle. Are your images reversed or not? If you watch a friend walk through the wall of a 3-dimensional Klein bottle, and look at all the images of her, which (if any) of them will be reversed?

(8) What is a good definition of left and right? Suppose that a person's back and front were identical, with feet and hands facing both directions and a face on both sides of her head. Now does the person have a distinct right and a left side?

(9) Why do we say a mirror reverses your left and right side, but not your head and feet? What if you were lying down or the mirror was on the ceiling?

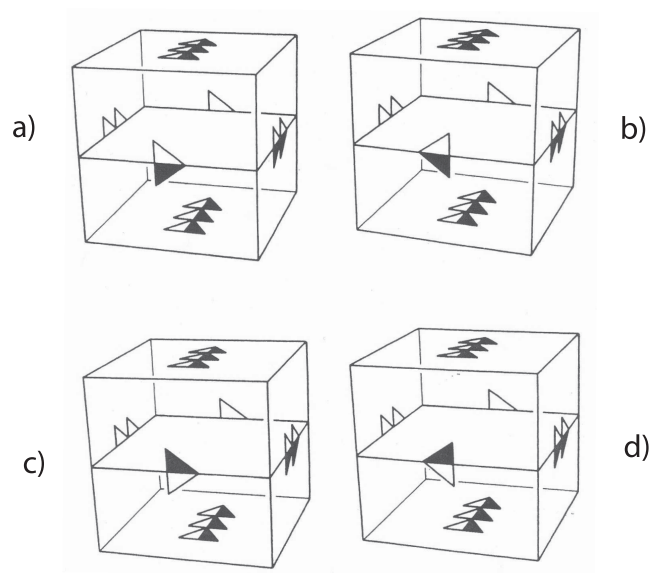
(10) Suppose A. Square lives alone in a small Klein bottle and each side of A. Square is colored a different color. Imagine that he can see in all directions, and his brain can determine relative distances. Draw an extended picture of A. Square's space

(11) List all three-in-a-row diagonals in Klein bottle tic-tac-toe?

(12) How many inequivalent first moves and second moves are there in Klein bottle tic-tac-toe? Recall that two moves are *equivalent* if the strategy for the rest of the game is analogous. This does not just mean that the NUMBER of ways to win is the same. Note that if there is a symmetry of the board then corresponding moves are equivalent.

(13) Will the first player necessarily win Klein bottle tic-tac-toe assuming optimal play? If so, list each of the moves the first player should make in order to win.

(14) Discuss the sidedness and orientability of the surfaces in the 3-manifolds in the Figure below.



(15) For each of the spaces the Figure above, draw a surface perpendicular to the given one and determine its sidedness and orientability.

Lecture 5

1. Connected Sums

In a homework problem we saw that two spaces can be joined together with an Einstein-Rosen bridge to create a new space. Whether the tube was long or short would have no effect on the topology. The tube was just a way to get from one of the spaces to the other. Now we will shorten the tube until it has no length at all, and it just becomes a circle. We will use the following method of joining spaces instead of using Einstein-Rosen bridges.

Definition 1. *Let A and B be surfaces. The* **connected sum**, *$A\#B$, is the surface obtained from A and B by removing a disk from each and gluing the remaining surfaces together along their boundary.*

For example, Figure 1 illustrates the connected sum of two tori, denoted by $T^2\#T^2 = 2T^2$.

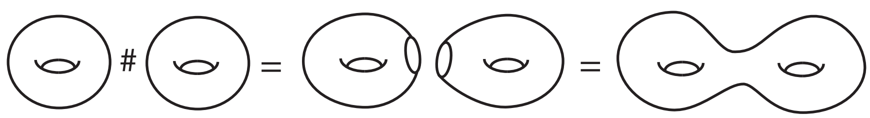


Figure 1. The connected sum of two tori.

For the moment we shall focus on the topology of connected sums and ignore the geometry. Thus we can slide the disk that will be removed around on A and B, and we can deform A and B to our heart's delight. Consider the connected sum of a torus T^2 and a sphere S^2. As we can see from Figure 2, $T^2\#S^2 = T^2$.

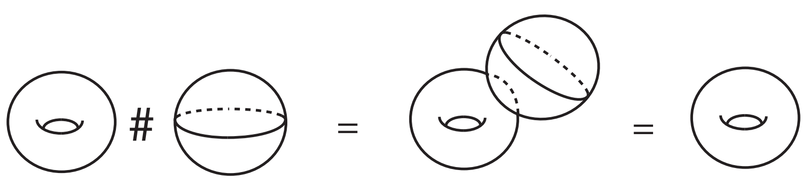


Figure 2. The connected sum of a torus and a sphere is a torus.

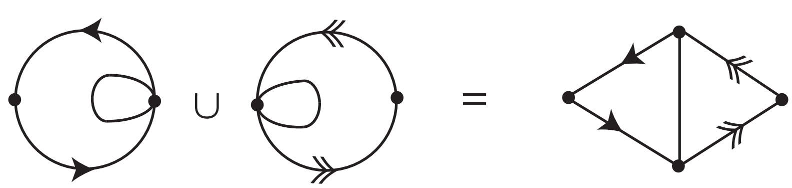


Figure 3. The connected sum of two projective planes.

Now consider the connected sum of two projective planes, $P^2 \# P^2 = 2P^2$ (see Figure 3) . To understand this surface we cut the surface apart and glue it together differently, keeping track of the original gluings. We see from the sequence of pictures in Figure 4 that $2P^2$ is a Klein bottle.

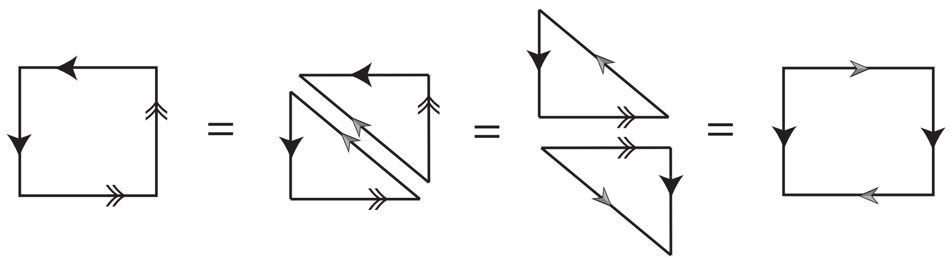


FIGURE 4. The connected sum of two projective planes is a Klein bottle.

Figure 5 gives us an idea of how we can represent n projective planes as a single glued up polygon. How many sides does a glued up polygon need to have to represent nP^2 in this way? Similarly we can represent $2T^2$ as in Figure 6. How many sides does a glued up polygon need to represent nT^2?

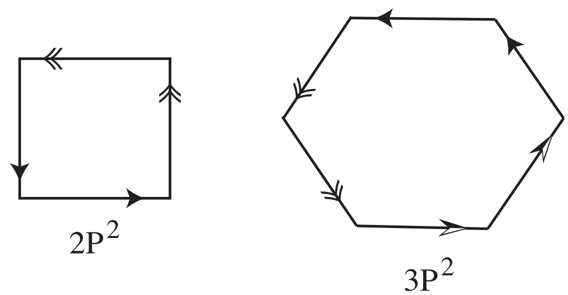


FIGURE 5. The connected sum of two and three projective planes.

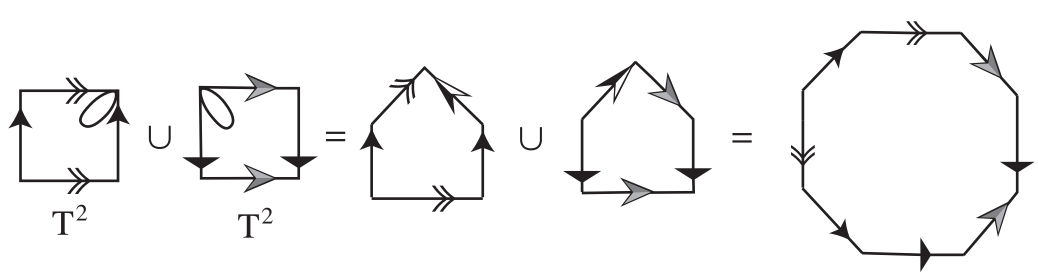


FIGURE 6. The connected sum of two tori.

The following is an important result in topology, which we won't prove.

Theorem. (The Classification of Surfaces) *Every 2-manifold with finite area is the connected sum of some number of tori and some number of projective planes, including $S^2 = 0T^2 \# 0P^2$.*

This theorem says that every finite surface has the form $nT^2 \# mP^2$. For homework you will prove that $T^2 \# P^2 = 3P^2$. From this equation and the Classification of Surfaces you will then prove that every finite 2-manifold has the form nT^2 or nP^2 (where n may be zero). This means we can get every surface without having to mix tori with projective planes. It follows that every orientable surface has the form nT^2 with $n \geq 0$, and every non-orientable surface has the form nP^2 with $n \geq 1$.

Example. $2T^2 \# P^2 = T^2 \# (T^2 \# P^2) = T^2 \# 3P^2 = (T^2 \# P^2) \# 2P^2 = 3P^2 \# 2P^2 = 5P^2$.

The idea of connected sums can also be extended to 3 dimensions.

Definition 2. *Let A and B be 3-manifolds. Then $A \# B$ is obtained by removing an open ball from each of A and B and gluing the remaining spaces together along their boundaries.*

Let T^3 denote the 3-torus, let P^3 denote the 3-dimensional projective plane, and let K^3 denote the 3-dimensional Klein bottle. We can use connected sums to create infinitely many 3-manifolds. For example, $T^3 \# T^3$, $T^3 \# P^3$, or $T^3 \# K^3$. How can we visualize these spaces?

2. Topological and Geometric Products

So far, the only finite volume 3-manifolds that we have seen are T^3, P^3, K^3 and connected sums of these. Now we introduce products as another way to create new spaces.

Definition 3. *Let X and Y be sets. Define $X \times Y = \{(x, y) | x \in X, y \in Y\}$.*

Thus every point in $X \times Y$ can be expressed uniquely as (x, y). For example, we are familiar with $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ where every point can be written uniquely as (x, y). We consider another example in Figure 7. Let S^1 denote a circle and let I denote the closed interval $[0, 1]$. Then $S^1 \times I$ is a cylinder, where every point on the cylinder has the form (x, y) with $x \in S^1$ and $y \in I$.

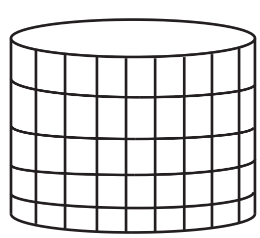


FIGURE 7. $S^1 \times I$ is a cylinder.

In general, (just like with multiplication of numbers) we can think of $X \times Y$ as an X of Y's and a Y of X's, where X and Y intersect at precisely one point (What would be the problem with them intersecting at more than one point?). So $S^1 \times I$ is both a circle of intervals and an interval of circles, as is illustrated in Figure 8.

FIGURE 8. $S^1 \times I$ is circle of intervals and an interval of circles.

Is a Möbius strip a product? It is a circle of line segments, but it is not a line segment of circles. In fact, all of the circles, except the central one meet the line segment twice (see Figure 9).

FIGURE 9. Is a Möbius strip a product?

So far we have only considered the topology of products, now we will consider the geometry of them as well.

Definition 4. *A product $X \times Y$ is said to be* **geometric** *if:*
1. *All of the X's are the same size.*
2. *All of the Y's are the same size.*
3. *All of the X's are perpendicular to all of the Y's.*

We would like our products to be geometric. Which of the spaces in Figure 10 are products? Which ones are geometric products?

A geometric product of two flat spaces (i.e, $\mathbb{R}$, S^1, I, T^2, or K^2) is itself flat. Hence triangles in such spaces have angle sums of 180°. Here are some examples of products whose local geometry is flat like that of Euclidean space: $S^1 \times S^1$, $S^1 \times I$, $S^1 \times \mathbb{R}$, $I \times I$, $T^2 \times I$, $S^1 \times S^1 \times S^1 = T^3$ $K^2 \times I$, $K^2 \times S^1 = K^3$, $\mathbb{R} \times T^2$, $\mathbb{R} \times K^2$. Some of these spaces are illustrated in Figure 11. What do the others look like?

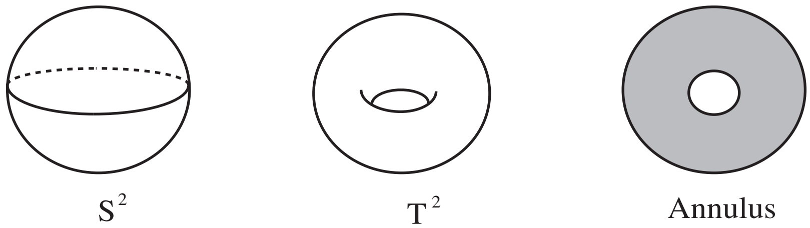


FIGURE 10. Which of these are products?

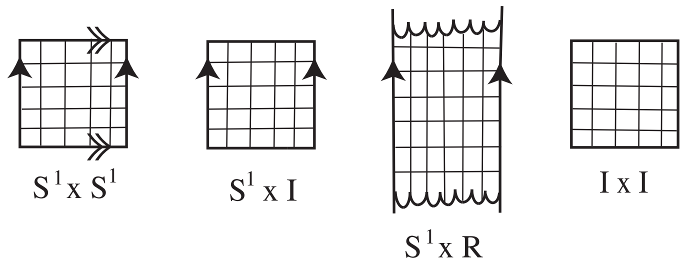


FIGURE 11. Some products with flat geometry

Some products whose local geometry is not flat are: $S^2 \times I$, $S^2 \times S^1$, $nT^2 \times I$, $nT^2 \times S^1$, $nP^2 \times I$, $nP^2 \times S^1$. Which of these and the above examples are 3-manifolds? By using products we can construct infinitely many different 3-manifolds, just as we could by using connected sums.

Flatlanders cannot visualize all 2-dimensional geometric products in a geometric way. For example, Flatlanders can't imagine a cylinder, so they think of $S^1 \times I$ as an annulus. However, in an annulus all of the circles are not the same size. So we have to tell the Flatlanders to imagine that all of the circles are the same size so that $S^1 \times I$ is actually a geometric product. When they draw $S^1 \times I$ as an annulus, rather than as a cylinder, it is distorted so that it is hard to see the geodesics. If two points are on the same circle on a cylinder the geodesic between them is also on that circle. But this is not obvious to the Flatlanders because of their distorted view (see Figure 12).

We have the same problem if we try to picture $S^2 \times I$. We can visualize it as a tennis ball (which is hollow inside). Such a tennis ball is an S^2 of intervals and an interval of S^2's. However, the spheres get smaller as you go towards the center of the ball. In order to make the product geometric, we have to follow the Flatlander's example and imagine that all of the spheres are the same size. Also for any pair of points on the same S^2, the geodesic between these points is also on the same S^2. If there are three points on a single S^2, then the triangle with those points as vertices is also contained in that S^2. Since this triangle is on a sphere, the sum of its angles is more than 180°, just as it is on any sphere (see Figure 13).

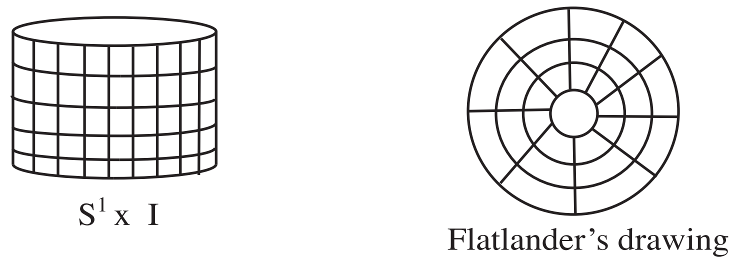


FIGURE 12. A geometric $S^1 \times I$ and a Flatlander's drawing of $S^1 \times I$

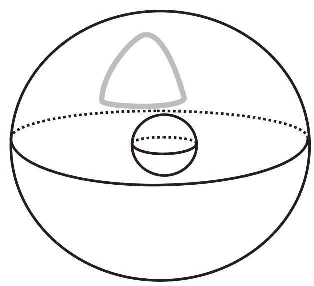


FIGURE 13. $S^2 \times I$ with a triangle contained in one of the S^2's

We can also consider a horizontal cross section of $S^2 \times I$ which goes through a great circle of every sphere (see Figure 14). This cross section is $S^1 \times I$. Since all the parallel spheres in a geometric $S^1 \times I$ are the same size, all the great circles in the cross section are the same size. Thus while this cross section looks to us like an annulus, it is actually a cylinder. Furthermore, for any pair of points in this cylindrical cross section, the geodesic between the points also lies in the cross section. Hence if there are three points in a single cross section, then the triangle with those points as vertices is also contained in that cross section. Thus the sum of the angles of a triangle with its vertices on this cross section of $S^2 \times I$ will always be $180°$.

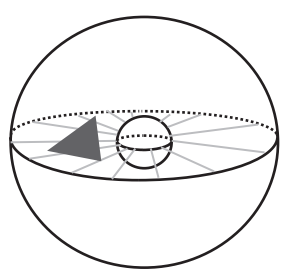


FIGURE 14. A flat cross section of $S^2 \times I$ with a triangle in it.

In summary, at any point on $S^2 \times I$, the local geometry is flat if you look in the direction of a $S^1 \times I$ cross section, and spherical if you look in the direction of a single S^2. So the geometry at a point depends on which direction you are looking.

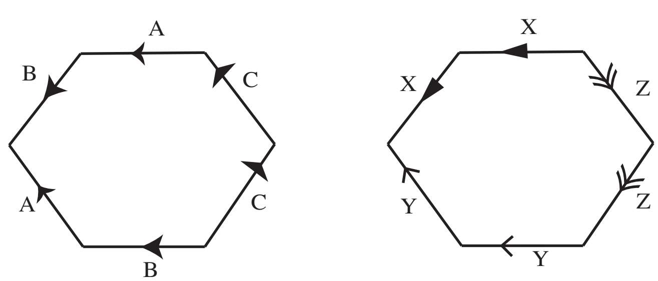
Definition 5. *A space is said to be* **isotropic** *if when you are standing at any point, the local geometry in a small enough region is the same in all directions.*

We have just seen that $S^2 \times I$ is not isotropic. All of the other spaces we have considered thus far are isotropic. On the other hand, $S^2 \times I$ is not a manifold because it has boundary. We would like to make a manifold whose local geometry is like that of $S^2 \times I$. The boundary of $S^2 \times I$ consists of an inner sphere and an outer sphere. To get rid of both boundary components we glue these two spheres together. Thus each interval, I, in the product now becomes a circle, S^1. In this way, we obtain the 3-manifold $S^2 \times S^1$. We have to glue the inner and outer spheres together abstractly, since it cannot be done in $\mathbb{R}^3$. Hence $S^2 \times S^1$ is a 3-manifold which is homogeneous but not isotropic.

Homework

(1) Is the connected sum operation commutative? Is it associative? Do surfaces have inverses with respect to the operation #?

(2) (Hard problem) Use cut and paste to prove that the two surfaces illustrated below have the same topology. Explain how this proves that $T^2 \# P^2 = P^2 \# P^2 \# P^2$.



(3) If we want to write the sum of n tori and m projective planes as a sum of some number of projective planes, how many projective planes must we use? Use induction together with the equation $T^2 \# P^2 = P^2 \# P^2 \# P^2$ to prove that your formula is correct (try to induct on only one variable). Why does this imply that every closed surface is either nT^2 or nP^2?

(4) Prove that T^2 can be obtained by gluing pairs of sides of any polygon with an even number of sides.

(5) Suppose that Flatland is a non-trivial connected sum. How could A. Square detect it topologically? Suppose our universe is a non-trivial connected sum. How could we detect it topologically?

(6) Find a manifold other than $S^2 \times S^1$ which is homogeneous but not isotropic?

(7) Find a manifold that is isotropic but not homogeneous?

(8) Suppose that we lived in a universe that we knew was one of $S^2 \times S^1$, $P^2 \times S^1$, or T^3. How could we tell the difference between these three spaces?

(9) Is the product operation distributive over the connected sum operation? That is, is $A \times (B\#C) = (A \times B)\#(A \times C)$?

Lecture 6

1. Flat manifolds

Recall the following definition:

Definition 1. *A* **flat manifold** *is a homogeneous space whose local geometry is Euclidean (i.e., the sum of the angles of every triangle is* $180°$*).*

So far the flat manifolds we have seen are T^2, K^2, T^3, K^3, $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R} \times S^1$, $\mathbb{R} \times T^2$, $\mathbb{R} \times K^2$. We have seen that a cone is flat everywhere except at the conepoint, where there around which are less than $360°$. A glued up cube is flat if it is homogeneous on the edges and the corners. Inside of the cube we already know the geometry is Euclidean. Thus the geometry is also Euclidean along the glued up faces. In order to make sure that there are $360°$ around every point on the edges and corners of the glued up cube, we have to check that all 8 corners are glued together and the edges are glued in groups of 4. Consider the K^3, which is illustrated in Figure 1. We can check that this manifold is indeed flat by coloring the groups of edges which are glued together with the same color and coloring the groups of corners which are glued together with the same color.

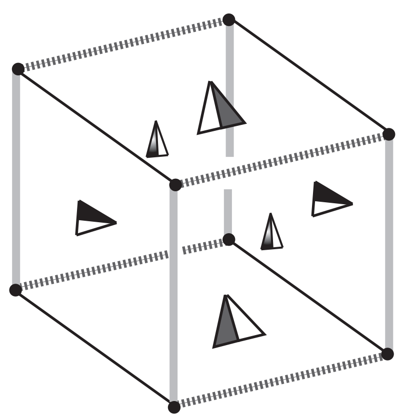


Figure 1. This construction of K^3 has homogeneous geometry because all 8 corners are glued together and the edges are glued in groups of 4.

One way to help us imagine what it would be like to live in K^3 is to draw an extended picture of K^3 (see Figure 2). A. Cube lives alone in this space and has different colors on each side, as indicated with letters (B=blue, O=orange, R=red, G=green, Y=yellow, and V=violet). Which pairs of faces are glued with a flip and which are glued straight? How are the flipped ones glued?

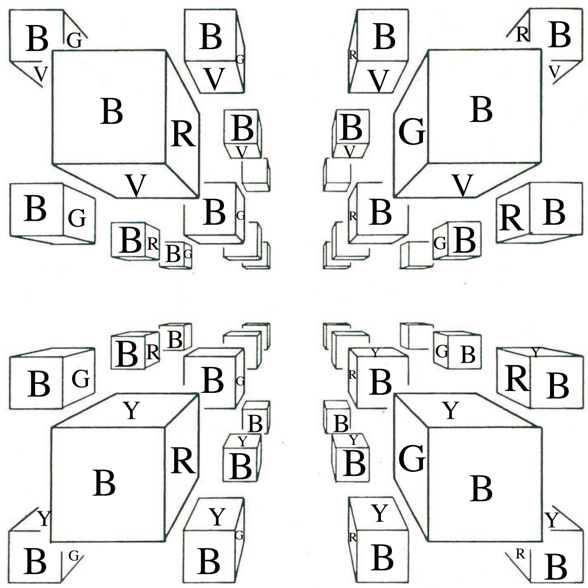


FIGURE 2. An extended diagram of K^3 with A. Cube in it.

2. Turn Manifolds

We can glue the sides of a cube together in various ways. But we always have to make sure the glued-up manifold is homogeneous by checking that that all 8 corners are glued together and the edges are glued in groups of 4.

One way to glue a pair of faces of a cube is with a rotation rather than straight or with a flip. For example, we create a $\frac{1}{4}$-*turn* manifold as follows. We glue the top of a cube to the bottom of the cube with a 90° rotation, while gluing the other sides straight. This means that when you go out the top you come in the bottom rotated by 90° (see Figure 3). Is this manifold homogeneous? Is it orientable?

Similarly, we could create $\frac{1}{2}$-turn and $\frac{3}{4}$-turn manifolds. Observe that such turn manifolds don't exist in 2-dimensions since if you start with a polygon the edges can only be glued straight or with a flip.

3. Conepoints

In any homogeneous manifold there are 360° around any point. If a manifold is homogeneous except at a few isolated points, we want to be able to talk about these bad points.

Definition 2. *If the angle sum around a point is less than* 360°*, then the point is said to be a* **cone point**. *If the angle sum around a point is more than* 360°*, then the point is said to be an* **anti-cone point**.

A cone point looks like the point on a cone. We can create a cone point of any angle by removing an appropriate angle from a disk and gluing up the

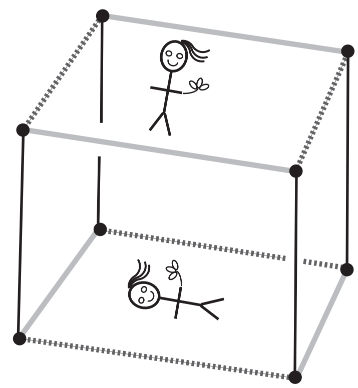


FIGURE 3. A $\frac{1}{4}$-turn manifold.

sides. See Figure 4. By contrast, there is too much surface around an anti-cone point. We can achieve this by cutting open a 360° angle and adding an extra angle.

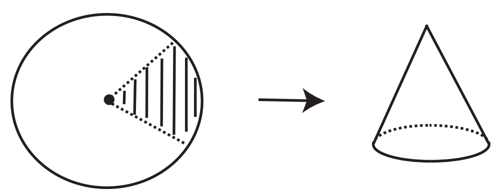


FIGURE 4. We can create a cone point by removing an angle from a disk and gluing up the new edges.

Homework

(1) Can you create a glued up cube where the edges are glued in groups of 4 but not all 8 vertices are glued together?

(2) Describe a 2-manifold that is homogeneous at all but five points. Find a 2-manifold that is homogeneous at all but n points.

(3) Describe a 3-manifold that is homogeneous everywhere except at some finite collection of circles.

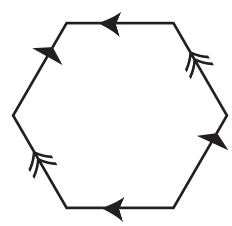
(4) Suppose that you live in a $\frac{1}{4}$-turn manifold and you speak on the phone to someone who lives in a $\frac{3}{4}$-turn manifold. Could you determine that your manifolds were different? What if you speak to someone in a $\frac{1}{2}$-turn manifold?

(5) Show that there are three different ways of gluing a cube such that two opposite pairs of walls are glued with a flip, and the remaining pair of walls is glue straight. Draw an extended diagram of A. Cube living in each of your spaces.

(6) Draw an octagon with edges glued in pairs to get a surface that has at least one cone point. Draw an octagon with a gluing that has at least one anti-cone point. Draw an octagon with a gluing that has at least one cone point and one anti-cone point. For each of your octagons indicate whether the space is orientable or non-orientable.

(7) For each $n > 1$, create nT^2 with a glued up polygon, then determine how many cone points and anti-cone points the surface has as a function of n. For each $n > 1$, create nP^2 with a glued up polygon, then determine how many cone points and anti-cone points the surface has as a function of n.

(8) What surface is obtained from the gluing illustrated below? Does this surface have cone points or anti-cone points?



(9) Suppose A Square lives in the surface illustrated above, and has a different color on each side. Draw an extended picture of A. Square's universe.

(10) Compare the global topology and geometry of a flat torus and the hexagonal gluing illustrated above.

(11) What surface do you get by gluing each side of a square to itself by folding the side in half. Does the surface have any cone points or anti-cone points? If so, how many?

(12) Start with a cube and glue each of the vertical sides to itself by folding the side in half along a floor to ceiling line. Then glue the floor of the cube to the ceiling with a $\frac{1}{4}$-turn. How many edges of non-homogeneity does this manifold have and what is the angle around each of these non-homogeneous edges?

(13) Consider the space obtained from a dodecahedron (a polyhedron whose sides are 12 identical pentagons) by gluing opposite faces with a one tenth turn. How many corners are glued together? How many edges are glued together?

(14) Suppose we live in a dodecahedron with opposite pairs of faces glued with a 1/10 turn. Can you tell the difference between this manifold and a dodecahedron with opposite pairs of faces glued with a 3/10 turn?

(15) Suppose that A. Square lives alone in Flatland. When A. Square looks around, how many images of himself does he see if Flatland is each of the following. Draw an extended picture in each case.
a) An infinitely long cylinder?
b) An infinite cone with cone angle 90°? c) An infinite cone with cone angle 73°? d) An infinite cone with cone angle 300°?

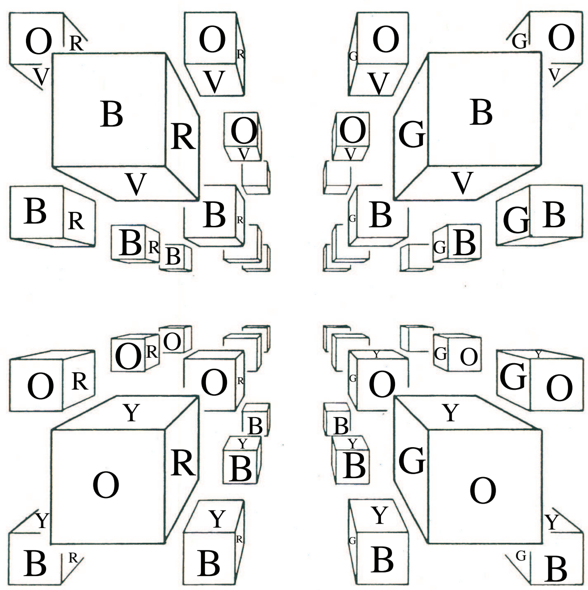


FIGURE 5. What gluings create this extended diagram?

(16) Explain the gluings in Figures 5, 6, and 7.

(17) Color Figure 8 to illustrate the view inside of a $\frac{1}{4}$-turn manifold.

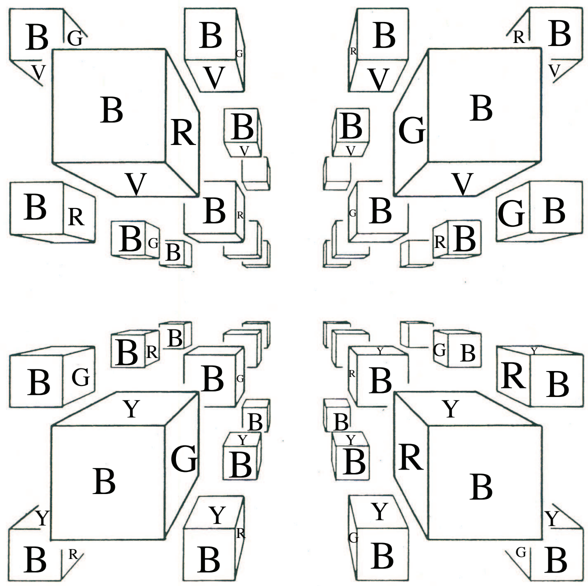


FIGURE 6. What gluings create this extended diagram?

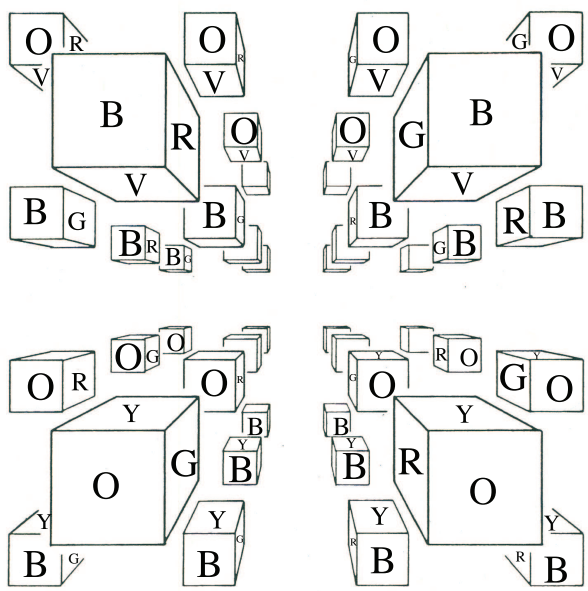


FIGURE 7. What gluings create this extended diagram?

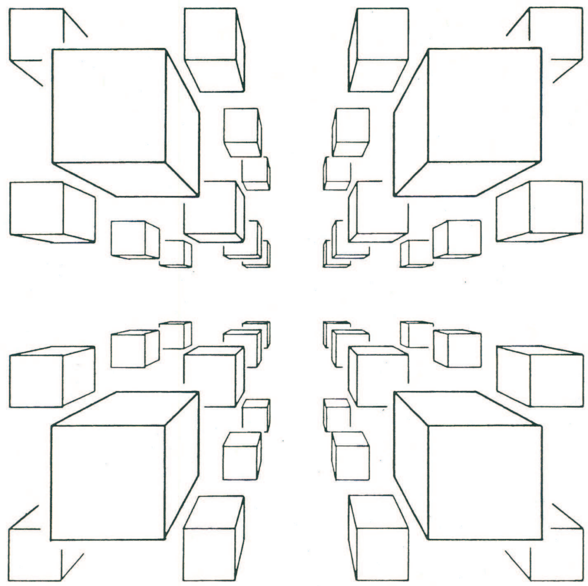


FIGURE 8. Color this extended diagram to illustrate the view inside of a $\frac{1}{4}$-turn manifold.

Lecture 7

1. Homogeneous Geometries

There are three types of local geometries for surfaces which are homogeneous. In each there are 360° around every point. However, the sum of the angles of triangles is different in each of the three geometries. Here are the types of geometries:

1) **Flat** or **Euclidean** geometry. In flat surfaces the sum of the angles of any triangle is 180°. These surfaces are said to have *zero curvature.*

2) **Elliptical** or **spherical** geometry. In elliptical surfaces the sum of the angles of any triangle is greater than 180°. These surfaces are said to have *positive curvature.*

3) **Hyperbolic** geometry. In hyperbolic surfaces the sum of the angles of any triangle is less than 180°. These surfaces are said to have *negative curvature.*

To help us understand hyperbolic geometry we consider the *hyperbolic plane* illustrated in Figure 1. Topologically the hyperbolic plane is the same as a regular plane, but geometrically it is like a saddle everywhere. Triangles on a hyperbolic plane have angle sums which are less than 180°. The bigger the triangle is the smaller it's angle sum will be. Also, a circle of a given radius has more area than a circle of the same radius in the plane.

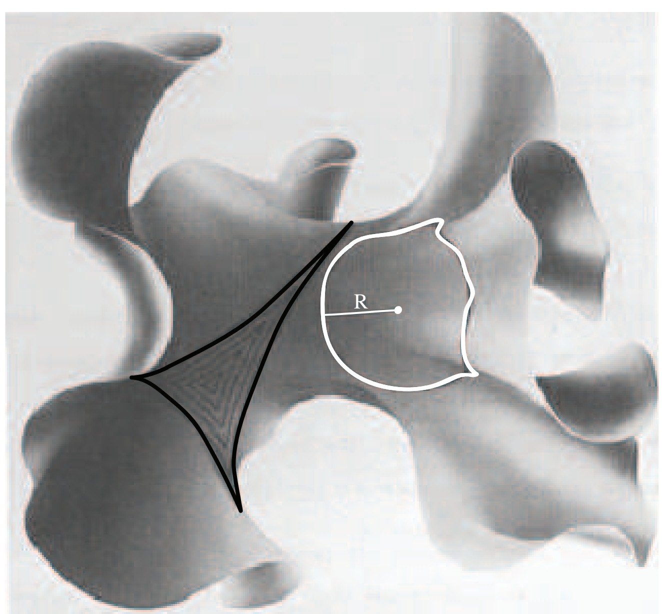


FIGURE 1. A hyperbolic plane with a triangle and a circle in it.

We would like all finite area 2-manifolds to have one of the homogeneous geometries. We know that every such surface is either a sphere, a connected sum of tori, or a connected sum of projective planes. We have seen that T^2 and $K^2 = 2P^2$ have flat geometry, and S^2 and P^2 have spherical geometry. We want to create homogeneous versions of nT^2 and nP^2 for other values of n. The key idea is that a glued up polygon does not have to be drawn on a flat piece of paper. It could be drawn on the surface of a sphere or on a hyperbolic plane. For example, we saw that if P^2 is drawn as a glued-up disk in the plane then it won't be homogeneous on it's boundary, but if it is drawn on a hemisphere than it will have spherical geometry.

We begin by considering $2T^2$, which can be created as a glued up octagon. However, all of the vertices of the octagon are glued together (see Figure 2). This glued-up point has more than 360° around it, so it is an anti-cone point.

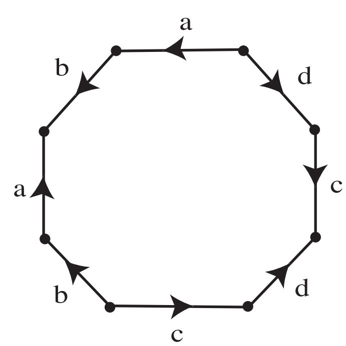


FIGURE 2. All of the vertices of this glued up octagon are glued together

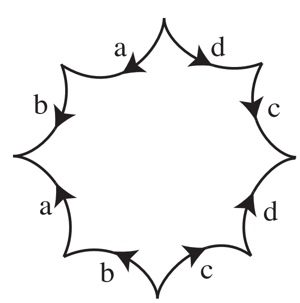


FIGURE 3. $2T^2$ made of a glued up hyperbolic octagon

It order to make the geometry homogeneous, we need to construct the octagon so that at the vertices the angles are $\frac{2\pi}{8}$. Then gluing all 8 vertices together would give us an angle of $2\pi = 360°$ at the glued up point, so we would no longer have an anti-cone point. To make the angles of the octagon smaller than they are on a flat plane, we draw the octagon on a hyperbolic plane. Since the sides of the octagon are made of hyperbolic geodesics, the angles between the sides are smaller than they were on the plane (see Figure 3). Furthermore, the bigger we draw the octagon on a hyperbolic plane, the smaller the angles will be. So we can choose to draw it exactly the right size so that each of the angles is precisely $\frac{2\pi}{8}$. This glued-up octagon is now $2T^2$ with homogeneous hyperbolic geometry.

For any $n \geq 2$, an nT^2 is made of a glued up $4n$-gon with all of its vertices glued together. We can create a homogeneous nT^2 with hyperbolic geometry by gluing up a hyperbolic $4n$-gon which has corner angles of exactly $\frac{2\pi}{4n}$. In this way the glued-up corner point will have an angle of 2π in the nT^2. This glued-up $4n$-gon will have homogeneous hyperbolic geometry.

For homework you will construct an nP^2 with homogeneous geometry for all $n > 2$. Thus we have a complete list of all finite area 2-manifolds, and each one has a homogeneous geometry. This means 2-manifolds are wonderful.

2. The 3-sphere

An important 3-manifold which we have not yet studied is the 3-dimensional sphere, S^3. We several ways to think about S^3 which are analogous to ways that we can think about S^2. The first approach is to consider $S^2 = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$ as the unit sphere in $\mathbb{R}^3$. By analogy, $S^3 = \{(x, y, z, w)|x^2 + y^2 + z^2 + w^2 = 1\}$ is the unit sphere in $\mathbb{R}^4$. This is a good analytical approach, but it doesn't give us much intuition about how to visualize S^3.

A second approach is to think of S^2 as two hemispheres which have been glued together. Topologically, each hemisphere is the same as a disk. So we can think of S^2 as two disks which have been glued together along their boundary (see Figure 4).

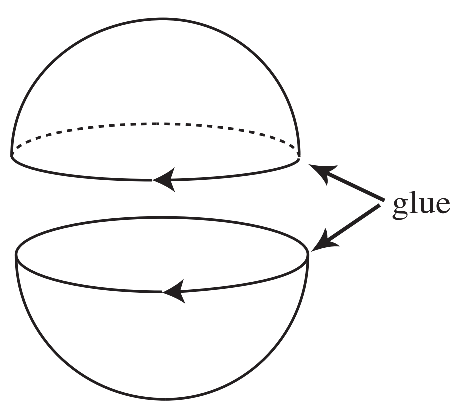


Figure 4. S^2 is made of two hemispheres which are glued together along their boundary

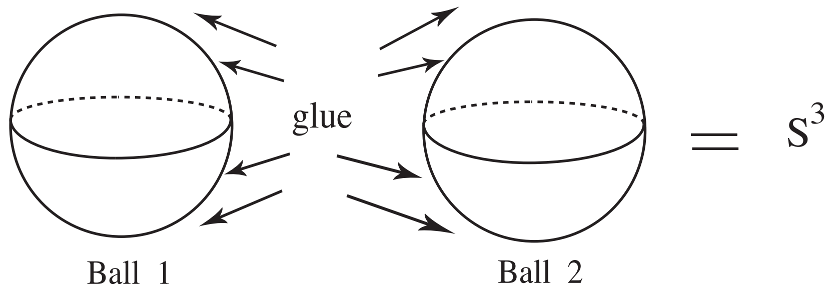


FIGURE 5. S^3 is made of two balls which are glued together along their boundary

Analogously we can divide S^3 into two "hemi-3-spheres." The 3-dimensional analog of a disk is a ball. So topologically each hemi-3-sphere is a ball. Thus we can think of S^3 as two balls which have been glued together along their boundary (see Figure 5).

A third approach is to consider S^2 with the North pole removed. We extend a line from the north pole through a point on S^2 to a point on the sphere (see Figure 6). This give us a bijective continuous map from S^2 with the North pole removed to $\mathbb{R}^2$. We can map the North pole itself to a point at "∞", which you get to if you go infinitely far in any direction. In this way we can consider $S^2 = \mathbb{R}^2 \cup \infty$.

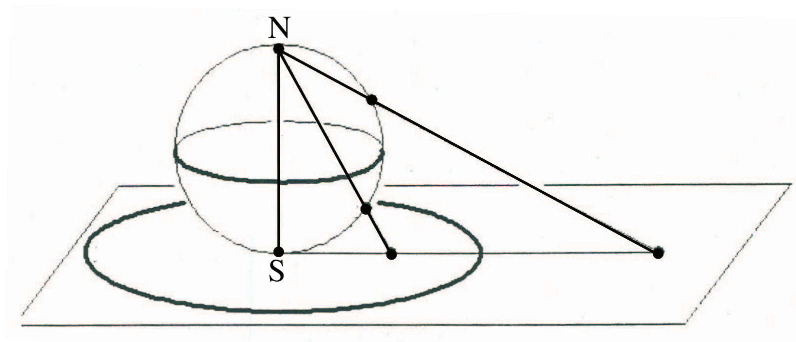


FIGURE 6. S^2 with the North pole removed is topologically equivalent to $\mathbb{R}^2$

Analogously, we can consider $S^3 = \mathbb{R}^3 \cup \infty$. That is S^3 is $\mathbb{R}^3$ together with a point at ∞ which we get to if we go infinitely far in any direction. This way of visualizing S^3 is convenient because it enables us to see that $\mathbb{R}^3$ is contained in S^3. Thus it is easy to go back and forth between thinking about $\mathbb{R}^3$ and thinking about S^3.

These are all ways to visualize S^3 topologically, but they do not tell us anything about the geometry of S^3. The geometry of S^3 is spherical (like

S^2), so the sum of the angles of any triangle in S^3 is greater than 180°. In fact, S^3 is a possibility for our own universe.

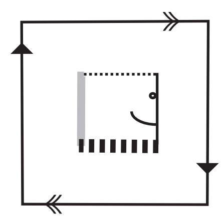
Homework

(1) Explain how to create an nP^2 with hyperbolic geometry for any $n > 2$. Explain why this method won't work for $n = 1$ or $n = 2$.

(2) Suppose that we glue opposite faces of a cube in each of the following ways. Is it possible to get a homogenous manifold? Explain.
a) One pair of sides is glued with a flip, and the other two pairs are each glued with a 180° rotation.
b) One pair of sides is glued with a flip, and the other two pairs are each glued with a 90° rotation.
c) Each pair is glued with a flip.

(3) Imagine a large square with a smaller square cut out of it. Glue together opposite sides of each square. What shape do you get?

(4) Suppose A. Square lives alone in the following space. He has a different color on each of his sides. He can see in all directions and has depth perception. Draw an extended picture of his space. Is his space homogeneous?



(5) Use the parallel postulate of Euclidean geometry (i.e. if a line crosses two parallel lines then the interior angles are equal) to prove that the sum of the angles of any flat triangle is π radians.

(6) Prove that the sum of the angles of a flat n-gon is $(n-2)\pi$ radians.

(7) Consider a unit sphere. What is the area of the double wedge created by two great circles where the wedge angles are $\frac{\pi}{3}$? Justify your conclusion.

(8) Let $L(x)$ be the area of the double wedge on a unit sphere created by two great circles whose wedge angle is equal to x radians. Find a formula for $L(x)$ in terms of x. Justify your conclusion.

(9) Let T be a triangle on the unit sphere with angles a, b, and c. Write a formula for the area of T in terms of $L(a)$, $L(b)$, and $L(c)$. Use it to show that the area of T is $a + b + c - \pi$. Note this implies that bigger triangles on the sphere have larger angle sums.

(10) Given that the area of a triangle T on a unit sphere is $a + b + c - \pi$, find a formula for the area of a similar triangle on a sphere of radius r. Justify your formula.

(11) Prove that the area of an n-gon on a unit sphere is $\theta_1 + \cdots + \theta_n - (n-2)\pi$, where the angles of the n-gon are $\theta_1, \ldots, \theta_n$.

(12) Divide a unit sphere into polygonal regions letting f denote the total number of regions, letting e denote the total number of edges, and letting v denote the total number of vertices. Use the result of the above problem to prove that no matter how the unit sphere is divided up into polygons, we always get the equation $v - e + f = 2$.

Lecture 8

Lectures 8 - 10 are based on *The Knot Book*, by Colin Adams

1. Knots, links, and projections

Knot theory is a subfield of topology whose broad goal is to develop methods to determine for any given pair of knots whether or not one can be deformed to the other. A simpler, yet still unsolved problem, is to find an algorithm that will to determine for any knot whether or not it can be deformed into a plane. We begin with some definitions.

Definition 1. *A* **knot** *is an embedding (i.e., a positioning) of a circle in* $\mathbb{R}^3$. *A* **link** *is an embedding of one or more disjoint circles in* $\mathbb{R}^3$. *Each circle in a link is called a* **component** *of the link.*

Notice that according to this definition, a knot is a link with a single component. So when we use the word "link" we could mean either a knot or a link with multiple components.

Definition 2. *A* **projection** *of a link is a drawing of it where we indicate undercrossings with gaps in the arcs.*

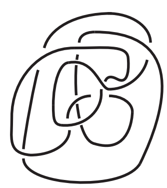


Figure 1. A projection of an unknotted circle

Definition 3. *An* **unknot**, **unlink**, *or* **trivial** *knot or link is one which can be deformed so that it lies in a plane with no self-intersections. We say a knot or link is* **non-trivial** *if it cannot be deformed into a plane.*

For example, Figure 1 is a projection of an unknot.

Definition 4. *Two knots or links are said to be* **equivalent** *if one can be deformed to the other.*

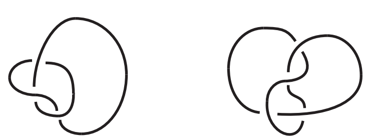


Figure 2. These two knots are equivalent.

We often are sloppy with notation and say that two knots are the same when we actually mean they are equivalent.

Just as we used the connected sum operation to build complicated manifolds from simpler ones, we can define the connected sum of knots in order to build complicated knots from simpler ones.

Definition 5. *Let P be a plane in $\mathbb{R}^3$, and let A be an arc in P. Suppose that K_1 and K_2 are knots on opposite sides of the plane P such that $K_1 \cap P = A = K_2 \cap P$. We define the* **connected sum** *of K_1 and K_2 as $K_1 \# K_2 = (K_1 \cup K_2) - A$ (see Figure 3).*

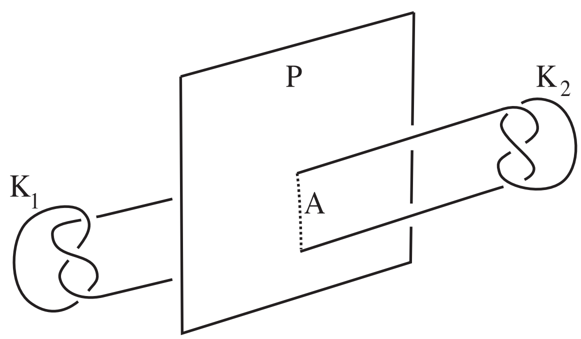


FIGURE 3. $K_1 \# K_2$

Note that this definition is a 1-dimensional version of the definition we gave earlier of the connected sum of two 2-manifolds and 3-manifolds, except that in this case the extrinsic topology is as important as the intrinsic topology. As with manifolds, we can use the connected sum operation to create infinitely many different knots.

Definition 6. *A knot which is the connected sum of two non-trivial knots is said to be* **composite**. *A knot which is not composite is said to be* **prime**.

Every knot is either prime or a connected sum of prime knots, just like every integer is either prime or a product of primes. We study prime knots as a way of understanding all knots, just as we study prime numbers as a way of understanding all numbers. For example, the trefoil knot illustrated in Figure 4 is prime because a plane cannot split it into two non-trivial knots.

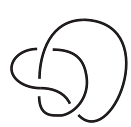


FIGURE 4. The trefoil knot is prime.

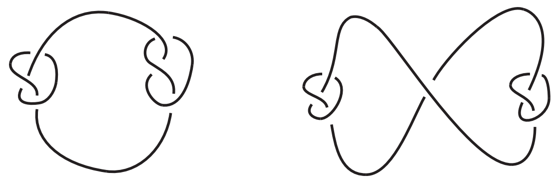


FIGURE 5. There are two different ways of creating a connected sum of two knots.

There are two different ways of creating a connected sum of a pair of knots K_1 and K_2 depending on how the ends of the two knots are glued together (see Figure 5). In Figure 5 the two ways of connecting the knots give equivalent connected sums. However, not all knots are the same when they are turned over. To keep track of when a knot is turned over we can put an arrow on the knot. A knot with an arrow on it is said to be *oriented* and the direction of the arrow is the knot's *orientation.* Figure 6 shows that when the trefoil knot is turned over, we get exactly the same projection that we started with, with the arrow reversed. The connected sums in Figure 5 are equivalent because the knot is unchanged when we turn it over, so it doesn't matter how the ends are connected.

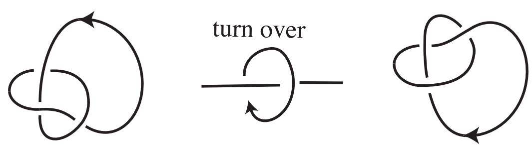


FIGURE 6. When we turn over the trefoil it is unchanged.

Definition 7. *If an oriented knot can be deformed so that its orientation is reversed, then we say the knot is* **invertible**. *If there is no such deformation, then we say the knot is* **non-invertible**

There are tables of every prime knot with up to 16 crossings. You can see a knot table for yourself at:

http://www.math.toronto.edu/ drorbn/KAtlas/Knots/

The fewest number of crossings that a knot K can be drawn with is called the *crossing number* of K and written $c(K)$. The knots in the tables are listed in order of crossing number. The trefoil knot is listed as 3_1 in the table. The 3 indicates that 3 is the fewest number of crossings of any projection of the knot. The 1 indicates that the trefoil is the first knot listed which has three crossings. In fact, there is no other non-trivial knot that can be drawn with three crossings. The mirror images of knots are not listed separately. So for example, the trefoil knot is listed in the tables, but its mirror image is not listed (its mirror image is illustrated in Figure 7). No knot in the table is equivalent to any other knot in the table or to the mirror image of any knot in the table.

FIGURE 7. The trefoil and its mirror image.

Most knots that do not have many crossings are invertible. The simplest knot which is non-invertible is 8_{17}. As with the trefoil, the 8 indicates that 8 is the fewest number of crossings of any projection of the knot. The 17 indicates that this knot is the seventeenth knot in the table which has 8 crossings.

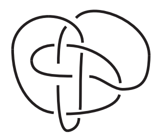


FIGURE 8. 8_{17} is the first non-invertible knot listed in the tables.

If K is an oriented knot then $-K$ denotes K with the opposite orientation, and K^* denotes the mirror image of K. The mirror image of a knot projection is obtained by switching all of the crossings from over to under and from under to over. If we are taking the connected sum of non-invertible knots we need to put orientations on the knots first. When we take the connected sum we connect the ends of the two knots in such a way that their orientations are consistent. In Figure 9 we illustrate $8_{17}\#8_{17}$, and in Figure 10 we illustrate $8_{17}\#-8_{17}$. Using this notation we saw above that $3_1\#3_1$ is equivalent to $3_1\#-3_1$. In the homework you will explore connected sums of K, $-K$, and K^*.

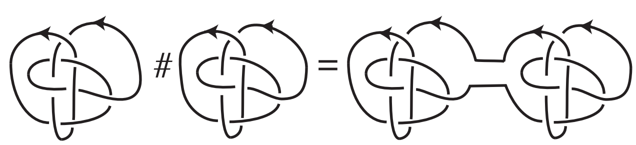


FIGURE 9. $8_{17}\#8_{17}$.

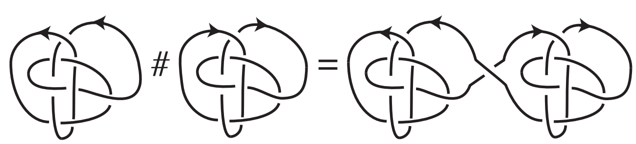


FIGURE 10. $8_{17}\#-8_{17}$.

2. Link Invariants

To show that two links are equivalent we only have to produce a deformation from one to the other. But to prove that two links are not equivalent, we would have to show that there can be no such deformation. This might seem impossible. However, it can be easy with the help of "link invariants".

Definition 8. *A knot or link* **invariant** *is a function f from the set of all knots or links to a set of mathematical objects (e.g., numbers, polynomials, groups, etc.) such that if L and L' are equivalent then $f(L) = f(L')$.*

For example, let's consider $f : \{links\} \to \mathbb{N}$ where $f(L)$ is the number of components (i.e., the number of circles) of L. If L is equivalent to L', then $f(L) = f(L')$. Hence f is a link invariant. Suppose we want to compare the *Hopf link* (illustrated in Figure 11) to the trefoil knot. We know that $f(3_1) = 1$ and $f(\text{Hopf}) = 2$. Hence the trefoil is not equivalent to the Hopf link. On the other hand, $f(4_1) = 1 = f(3_1)$, where 4_1 denotes the figure eight knot. But this does not imply that the figure eight knot is equivalent to the trefoil. So this invariant, does not allow as to distinguish different knots

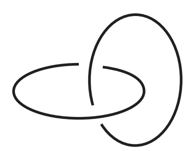


Figure 11. The Hopf link.

The idea of an invariant is that to each link we assign a mathematical value which is simpler than the link itself. If two links are assigned different values then we know that the links are not equivalent. If two links are assigned the same value, it doesn't tell us anything (as we saw in the example above). No known link invariant distinguishes every pair of inequivalent links. So we may have to try several invariants before we are able to distinguish two links or two knots. We will introduce more invariants after some definitions. Remember that an *oriented link*, is a link with an arrow on each component.

Figure 12. A negative (left-handed) crossing and a positive (right-handed) crossing.

Definition 9. *Figure 12 illustrates two types of crossings that may occur in an oriented link projection. The crossing on the left side is called a* **left-handed** *crossing or a* **negative** *crossing, and the crossing on the right side is called a* **right-handed** *crossing or a* **positive** *crossing.*

You can tell whether a crossing is left-handed as follows. If you point your left thumb in the direction of the overcrossing arrow and you curl your fingers, then your fingers point in the direction of the undercrossing arrow. By contrast, if you point your right thumb in the direction of the overcrossing of a left-handed crossing, your fingers cannot curl in the direction of the undercrossing arrow. Using your right hand you can similarly recognize a right-handed arrow.

Consider the oriented projection of the trefoil knot in Figure 13. You should check that this knot has three negative crossings and no positive crossings.

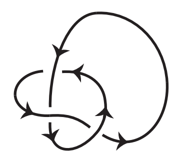


FIGURE 13. This oriented projection of a trefoil knot has three negative crossings.

We will use positive and negative crossings to define another type of link invariant. First we need the following definition.

Definition 10. *The* **writhe** *of an oriented link projection $w(L)$ is the total number of positive crossings minus the total number of negative crossings of the projection.*

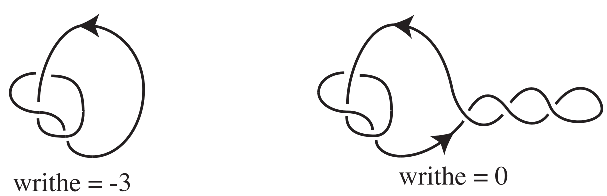


FIGURE 14. These two trefoil knot projections have different writhes.

In Figure 14 we illustrate two different projections of the trefoil knot, one of which has writhe -3 and the other of which has writhe 0. Since these are equivalent knots which have different writhes, we see that the writhe is not a link invariant. In fact, by adding kinks to a given knot we can create an equivalent knot with any writhe we want. However, if we only count crossings between different components of a link, then adding kinks has no effect on the sum. This motivates us to make the following definition.

Definition 11. *Consider an oriented link projection with components* K_1 *and* K_2*. We define the* **linking number** *of* K_1 *and* K_2*, denoted by* $\mathrm{lk}(K_1, K_2)$*, as* $\frac{1}{2}$ *the number of positive crossings between* K_1 *and* K_2 *minus the number of negative crossings between* K_1 *and* K_2*.*

For example, in Figure 15, the link on the left has $\mathrm{lk}(K_1, K_2) = 1$ and the link on the right has $\mathrm{lk}(K_1, K_2) = 0$. The linking number is a link invariant (as you will prove in the homework). So we know that the links in Figure 15 are not equivalent.

lk(K1, K2) = 1   lk (K1, K2) = 0

FIGURE 15. The link on the left has $\mathrm{lk}(K_1, K_2) = 1$ and the link on the right has $\mathrm{lk}(K_1, K_2) = 0$.

3. Reidemeister moves

We would like to prove that the linking number is a link invariant. In order to do this we introduce an important technique developed by the German mathematician Kurt Reidemeister.

Observe that a knot projection is made up of arcs in the plane. By deforming an arc of the projection within the plane keeping its endpoints fixed we will get the projection of an equivalent knot (see Figure 16).

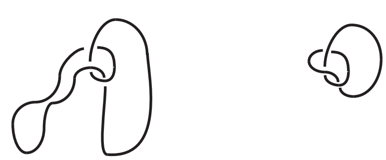


FIGURE 16. A deformation of an arc in the projection takes the knot on the left to the one on the right.

In 1926, Reidemeister proved that every deformation of a knot or a link in space can be achieved by planar deformations fixing the endpoints of arcs together with the following three *Reidemeister moves*.

(R1) A kink can be added or removed as in Figure 17.

(R2) An arc can be slid over or under another arc, keeping all endpoints fixed as in Figure 18.

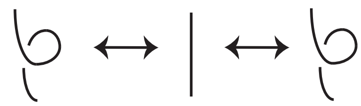


FIGURE 17. (R1) A kink can be added or removed.

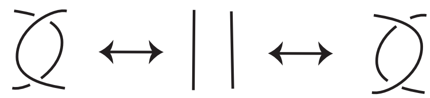


FIGURE 18. (R2) An arc can be slid over or under another arc.

(R3) An arc can be slid over or under a crossing, keeping all endpoints fixed as in Figure 19.

FIGURE 19. (R3) An arc can be slid over or under a crossing.

It is not hard to check that each of these moves is a deformation of the knot or link. But it is hard to prove that every deformation of a knot or link can be achieved by a finite sequence of these moves. This is what Reidemeister proved. You should NEVER use Reidemeister moves to deform one link to another, because that makes the process slow and extremely painful. Rather, you should use Reidemeister moves to prove that a given property is a link invariant. In particular, if we can show that a property of a link does not change when these moves are applied, then we know that no matter how the link is deformed, then that property will remain the same. This shows that the property is a link invariant. In the homework you will use this strategy to prove that the linking number is an invariant.

Homework

(1) Prove that every knot has a projection with over 1000 crossings.

(2) Prove that the writhe is invariant under the moves R2 and R3, but not R1.

(3) Use the Reidemeister moves to prove that the linking number is an invariant of oriented links.

(4) (hard) Prove that the linking number is always an integer.

(5) For any n, prove that there are infinitely many non-equivalent oriented 2-component links with linking number n, each of whose components are unknotted.

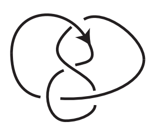
(6) Let L denote an oriented link with components J and K. Let $-J$ and $-K$ denote J and K, respectively with their orientations reversed. Let L^* denote the mirror image of L with components J^* and K^*. Suppose that $\text{lk}(J,K)=n$. Find $\text{lk}(-J,K)$, $\text{lk}(J,-K)$, $\text{lk}(-J,-K)$, and $\text{lk}(J^*,K^*)$. Justify your conclusions.

(7) Suppose that an oriented link with components J and K can be deformed to its mirror image preserving the orientation on each component. Prove that $\text{lk}(J,K)=0$.

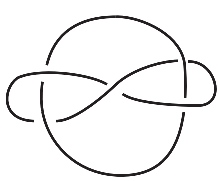
(8) Find an unoriented link with components J and K, which can be deformed to its mirror image, but once you give the link an orientation $\text{lk}(J,K)\neq 0$

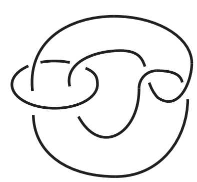
(9) Prove that for a given projection of a knot the writhe is independent of the orientation of the knot.

(10) To draw the mirror image of a projection of a knot or link you just switch every undercrossing to an overcrossing. Draw a deformation of an oriented figure eight knot (see below) to its mirror image that does not reverse the orientation on the knot. You may want to color the arcs of your knot with different colors to make it clear which part of the knot is being deformed at each step. Try to make your deformation short and easy to follow. In particular, do NOT use Reidemeister moves.



(11) Prove that the two links illustrated below are equivalent.





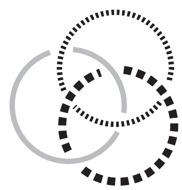
(12) Are the knots $8_{17}\#8_{17}$ or $8_{17}\#-8_{17}$ invertible?

(13) Is $3_1\#8_{17}$ equivalent to $3_1\#-8_{17}$? Is $3_1\#8_{17}$ invertible?

(14) If K_1 is invertible and K_2 is non-invertible, are $K_1\#K_2$ and $K_1\#-K_2$ equivalent? Is $K_1\#K_2$ necessarily invertible? Explain.

(15) Prove that any knot can be unknotted by changing some number of crossings. Hint: Think about how a water slide goes down and it never comes back up.

The link illustrated below is special because if you remove any component the link fails apart. In particular, the linking number between any pair of components is zero. This link is considered to be very beautiful and has even appeared in various works of art and architecture. More generally, we have the following definition.



Definition 12. *A link with n components, $L_1, \ldots, L_n$, is said to be* **Brunnian** *if the link is non-trivial, but removing any one component gives the trivial link.*

(16) Find Brunnian links with arbitrarily many components.

Lecture 9

1. Tricolorability

We now consider a different type of invariant.

Definition 1. *We say a projection of a link is* **tricolorable** *if there is a way to color each arc of the projection with one of three colors such that:*
1) More than one color is used.
2) At each crossing, either only one or all three colors occur (see Figure 1).

Figure 1. At each crossing there is either only one or all three colors.

We can see from Figure 2 that the usual projection of a trefoil knot is tricolorable.

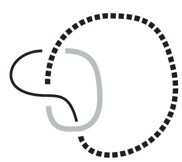


Figure 2. The trefoil knot is tricolorable.

Lemma. *The projection of a figure eight knot illustrated in Figure 3 is not tricolorable.*

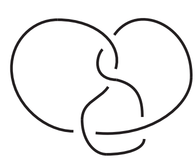


Figure 3. This knot projection is not tricolorable.

Proof. Suppose that there is a three coloring of the projection illustrated in Figure 3. We will obtain a contradiction as follows. We begin by considering the arc on the right, which is colored with some color, say black (see Figure 4).

FIGURE 4. We can assume the arc on the right is black.

Now we focus on the lower endpoint of this black arc. We have two cases according to whether there is only one color at this crossing or all three colors.

Case 1: The bottom crossing uses only one color (see Figure 5).

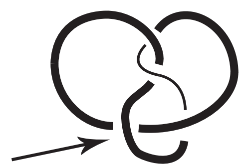


FIGURE 5. Case 1: All three arcs at the bottom crossing are black.

Now since two of the arcs at the top crossing are black, the third arc must also be black. As there is only one remaining arc in the projection, this means that the entire projection is black. But this violates Condition 1 of the definition of tricolorability.

Case 2: The bottom crossing uses all three colors, say black, grey, and dotted (see Figure 6).

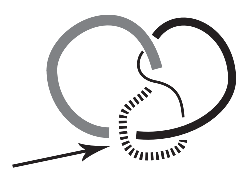


FIGURE 6. Case 2: The bottom crossing uses all three colors.

Now the top crossing is already colored with two colors, say black and grey. Thus the third arc at this crossing must be dotted. However this is the only remaining arc of the projection. But if this is dotted then the remaining crossings each use only two colors, which violates Condition 2 of the definition of tricolorability (see Figure 7). Hence our original projection of the figure eight knot was not tricolorable. □

For homework you will show that the Reidemeister moves preserve tricolorability. So either every projection of a knot or link is tricolorable or

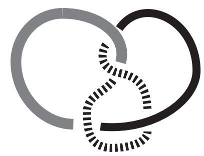


FIGURE 7. This coloring violates Condition 2 of the definition of tricolorability.

no projection is tricolorable. This means that whether or not a knot is tricolorable is an invariant, and hence is independent of the projection of the knot. In particular, by the above Lemma no projection of the figure eight knot is tricolorable. Also, since the trefoil knot has a tricolorable projection, we know that the figure eight is not equivalent to the trefoil. The unknot has a projection with no crossings, and hence is not tricolorable. Thus it also follows that the trefoil knot is not equivalent to the unknot. We don't yet know how to prove that the figure eight knot is not equivalent to the unknot.

2. Crossing Number

Recall that the *crossing number*, $c(K)$, of a knot or link K is the fewest number of crossings that it can be drawn with. In one of the homework problems you will prove that $c(K)$ is a link invariant. However, in general $c(K)$ is hard to compute. Just because a given projection of a knot has a certain number of crossings doesn't mean that there can't be a completely different projection of the same knot with fewer crossings. In fact the following is still an unproven conjecture.

Conjecture: For any knots K_1 and K_2, $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

For example, we can draw the connected sum of two trefoil knots so that it has six crossings. But how do we know that it could not be drawn in some other way with only five or even fewer crossings?

There is a special type of link which is simpler to analyze than others.

Definition 2. *An* **alternating** *link is one which has a projection that alternates between over and under crossings as you go around each component (see Figure 8).*

Definition 3. *A crossing in a projection is said to be* **nugatory** *if it looks like one of the crossings in Figure 9. There can be any number of crossings inside the black dot or outside of the black dot, but no other part of the link goes in or out of the black dot.*

The word nugatory refers to the fact that such a crossing contributes nothing to the link. For example, the knot illustrated in Figure 10 has a nugatory crossing in the center of the projection. As we can see from this

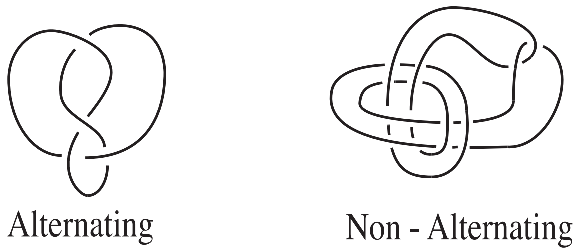


FIGURE 8. An alternating knot and a non-alternating knot.

FIGURE 9. A nugatory crossing looks like one of these.

example, a nugatory crossing can easily be eliminated by untwisting that part of the knot and this won't introduce any new crossings.

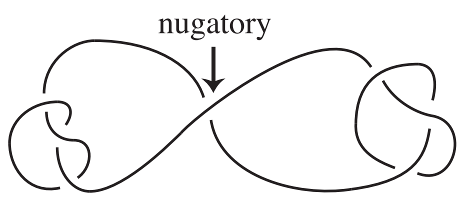


FIGURE 10. This knot has a nugatory crossing which can easily be removed.

Definition 4. *A projection of a link is said to be* **reduced** *if it contains no nugatory crossings.*

The following important result was proved by Kauffman, Murasugi, and Thistlethwaite.

Theorem. *A reduced alternating projection of a link has the smallest number of crossings of any projection of the link.*

This theorem makes it easy for us to recognize that certain projections have a minimal number of crossings. For example, we can easily check that the complicated knot projection in Figure 11 is reduced and alternating. Hence we immediately know that it has a minimal number of crossings. The theorem also implies that the usual drawing of the trefoil knot (with three crossings) and the usual drawing of the figure eight knot (with four crossings) each have a minimal number of crossings. In particular, this means that the figure eight cannot be deformed to the trefoil since the trefoil can be drawn with fewer crossings. In fact, we already knew that the trefoil and figure eight were not equivalent since the trefoil is tricolorable and the figure eight isn't. But now we also know that the figure eight cannot be deformed to an unknot, since the figure eight knot has no projection with fewer than four crossings.

FIGURE 11. This is a reduced alternating projection.

Thistlethwaite also proved the following interesting result about alternating links.

Theorem. *All reduced alternating projections of a given oriented link have the same writhe.*

It follows from this theorem that for alternating links, the writhe of a reduced alternating projection is a link invariant. This theorem allows us to prove that the knots in Figure 12 are not equivalent, even though they both have the same crossing number. First observe that both projections are reduced and alternating. We saw in a previous homework that the writhe of a knot projection is independent of the orientation of the knot. So we don't bother to put an arrow on the projection. The 6-crossing knot on the left has writhe equal to -2, while the 6-crossing knot on the right has writhe equal to 0. Thus the two knots cannot be equivalent.

FIGURE 12. These alternating knots are not equivalent since they have different writhes.

Homework

(1) Show that the Reidemeister moves (R3) preserve tricolorability. Make sure that your proof covers all cases.

(2) Consider all of the seven crossing knots in the tables and determine which ones are tricolorable.

(3) Let K_1 be a tricolorable knot and let K_2 be any knot. Prove that $K_1 \# K_2$ is tricolorable.

(4) Recall that $c(L)$ is the smallest number of crossings that any projection of a deformation of L can be drawn with. Prove that $c(L)$ is a link invariant.

(5) Prove that there are no non-trivial knots with only two crossings.

(6) How many non-equivalent 2-component links are there having a projection with two crossings? Justify your conclusion.

(7) Give an upper bound for the number of links there are with crossing number n (don't try to find the least upper bound).

(8) Prove that for all knots K_1 and K_2, $c(K_1 \# K_2) \leq c(K_1) + c(K_2)$.

(9) Let K_1 and K_2 be knots. Prove that if K_1 and K_2 are alternating then $c(K_1 \# K_2) = c(K_1) + c(K_2)$ Hint: first show that $K_1 \# K_2$ can be deformed to an alternating projection.

(10) Let L denote an oriented link projection. Let L^* denote the mirror image of L. How is the writhe of L^* related to the writhe of L? Can the writhe ever be used to prove that a link is not equivalent to its mirror image? Explain.

(11) Show that any knot projection can be changed to an alternating projection by changing a finite number of crossings. What is the maximum number of crossings that you have to change in an n-crossing projection to make it alternating? Explain.

Lecture 10

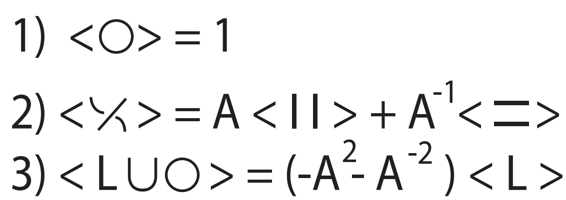
1. Knot Polynomials

So far the link invariants that we have studied are: number of components, crossing number, linking number, tricolorability, and writhe for reduced alternating projections. All of these invariants are numbers except for tricolorability, which has the values "yes" or "no".

Now we introduce a polynomial invariant. Polynomial invariants are more complicated than numerical invariants, but they enable us to distinguish a lot more knots and links. The polynomial invariant that we introduce is, strictly speaking, not a polynomial because it has negative as well as positive integer exponents. Such a function is called a *Laurent polynomial*, but it is common to be sloppy and just call it a polynomial.

We begin by defining the Kauffman *bracket* polynomial. This won't be an invariant, but we will use it to define our invariant. We will define the bracket polynomial recursively on a link projection. That is, first we define it on a planar circle (i.e. a circle with no crossings), then we use a set of rules to compute the bracket polynomial for a more complicated projection in terms of a simpler one.

Definition 1. *The* **bracket** *polynomial $\langle L\rangle$ of a link projection L is a Laurent polynomial with variable A, defined recursively by the three rules given below.*



Note when we put a little picture of part of a link projection inside of the bracket symbol, we mean that apart from that little picture the projection is the same as the original link projection. Rule 1) says that the bracket polynomial of a circle with no crossings is equal to 1. Rule 3) says that if we add a disjoint planar circle to an existing link projection then we multiply the bracket polynomial by $-A^2 - A^{-2}$. Using Rules 1) and 3) we can find the bracket polynomial of a planar unlink projection with more than one component. For example,

$$\langle \bigcirc \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle \bigcirc \rangle = -A^2 - A^{-2}$$

By induction the bracket polynomial of a planar projection with n components is $(-A^2 - A^{-2})^{n-1}$.

Rule 2) allows us to reduce any link projection to projections of two new links each with fewer crossings than the original. In this way, we can progressively reduce our link projection to a collection of simpler link projections, whose bracket polynomials we already know. The three projections in the equation for Rule 2) are identical outside of the single crossing that we are focusing on. We cut the overcrossing as illustrated in Figure 1, then attach the segments in two different ways.

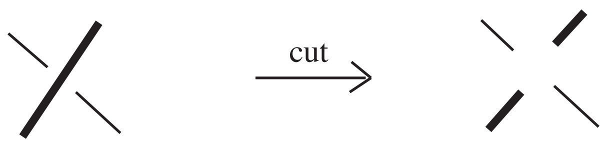


FIGURE 1. We cut the overcrossing.

Each endpoint of a bold arc (formerly part of an overcrossing) is now attached to an adjacent endpoint of a thin arc (formerly part of an undercrossing) . According to Rule 2) when we attach the bold arc to an adjacent arc in the clockwise direction we multiply the bracket polynomial of the new link by A, and when we attach the bold arc to an adjacent arc in the counterclockwise direction we multiple the bracket polynomial of the new link by A^{-1} (see Figure 2).

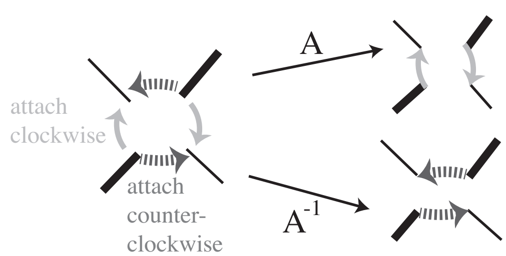


FIGURE 2. There are two ways to reattach the endpoints.

To better understand this process, we consider a couple of examples. First we compute the bracket polynomial of an unknot with a single positive crossing in Figure 3. Then in Figure 4 we compute the bracket polynomial of a Hopf link.

FIGURE 3. The bracket polynomial of an unknot with one crossing.

We will see as follows that the bracket polynomial is not a link invariant. We add the positive kink (illustrated in Figure 5) to a link projection L to

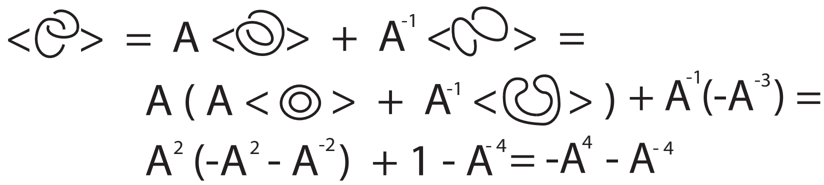


FIGURE 4. The bracket polynomial of a Hopf link.

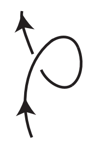


FIGURE 5. A positive kink.

get a new projection L'. Note that this kink is called "positive" because if we oriented it we would have a positive crossing.

In Figure 6 we compute the bracket polynomial of L' in terms of the bracket polynomial of L. We find that $\langle L' \rangle = \langle L \rangle(-A^3)$, and hence $\langle L' \rangle \neq \langle L \rangle$. Since L and L' are equivalent links, this shows that the bracket polynomial is not a link invariant. In particular, the bracket polynomial does not preserve Reidemeister move (R1). Similarly, we can add a negative kink to a projection of a link L to get a new link projection L' and show that $\langle L' \rangle = \langle L \rangle(-A^{-3})$. We will now use the bracket polynomial to define another Laurent polynomial which will actually be an invariant of oriented links.

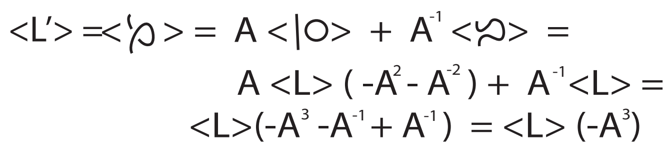


FIGURE 6. If L' is obtained from L by adding a positive kink then $\langle L' \rangle = \langle L \rangle(-A^3)$. If L' is obtained from L by adding a negative kink then $\langle L' \rangle = \langle L \rangle(-A^{-3})$.

Definition 2. *Let L be an oriented link projection and let $w(L)$ denote the writhe of L. Define the* **X-polynomial** *to be $X(L) = (-A^3)^{-w(L)} \langle L \rangle$.*

We did not need a link projection to be oriented in order to compute the bracket polynomial. However, since the writhe is only defined for oriented projections, $X(L)$ is only defined if L is an *oriented* projection. In order to better understand this definition, we compute the X-polynomial of the oriented Hopf link L illustrated in Figure 7

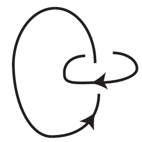


FIGURE 7. The oriented Hopf link L.

Recall that $\langle L \rangle = -A^4 - A^{-4}$. Also we see from Figure 7 that $w(L) = -2$. Thus we have

$$X(L) = (-A^3)^{-(-2)}(-A^4 - A^{-4}) = A^6(-A^4 - A^{-4}) = -A^{10} - A^2$$

Theorem. *The X-polynomial is an invariant of oriented links.*

Proof. Let L be an oriented link projection. You will prove on the homework that $\langle L \rangle$ is preserved by Reidemeister moves R2 and R3. Since $w(L)$ and $\langle L \rangle$ are both invariant under Reidemeister moves R2 and R3, $X(L)$ is also invariant under Reidemeister moves R2 and R3. So we just need to prove that $X(L)$ is invariant under Reidemeister move R1. Let L' denote L with a positive kink added. Then $w(L') = w(L) + 1$. Also we saw above that $\langle L' \rangle = \langle L \rangle(-A^3)$. Hence

$$X(L') = (-A^3)^{-w(L')}\langle L' \rangle = (-A^3)^{-w(L)-1}\langle L \rangle(-A^3) =$$
$$(-A^3)^{-w(L)}(-A^3)^{-1}\langle L \rangle(-A^3) = (-A^3)^{-w(L)}\langle L \rangle = X(L)$$

Thus the X-polynomial is invariant under adding or removing a positive kink. By an analogous argument, the X polynomial is invariant under adding or removing a negative kink. Thus the X-polynomial is invariant under all three Reidemeister moves, and hence is an invariant of oriented links. □

This polynomial invariant was originally discovered by Vaughan Jones in a different form. To get from the Jones polynomial V with variable q, to the X-polynomial with variable A, we use the substitution $q = A^{-4}$. The Jones polynomials of knots and links with up to 10 crossings are given in the knot tables. For example, under the trefoil knot is written the code $\{-4\}(-1101)$. The -4 indicates that -4 is the lowest exponent in the Jones polynomial, and the sequence of numbers represents the sequence of coefficients of the variables with increasing exponents. This means that the Jones polynomial of the trefoil knot is $V(3_1) = -q^{-4} + q^{-3} + q^{-1}$. Using the substitution $q = A^{-4}$ we find that the X polynomial of the trefoil knot is $X(3_1) = -A^{16} + A^{12} + A^4$.

Homework

(1) Prove that the bracket polynomial is preserved by Reidemeister moves R2 and R3.

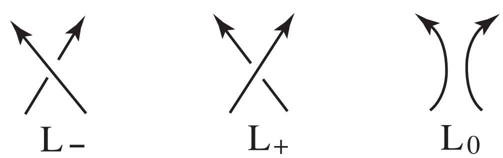
(2) If L is an oriented link, then $-L$ is the link with the orientation of every component reversed, and L^* is the mirror image of L. How is $X(L)$ related to $X(-L)$? How is $X(L*)$ related to X(L)? Justify your conclusions.

(3) Compute the X polynomial of the trefoil knot, then use problem 2 to show that the trefoil cannot be deformed to its mirror image.

(4) Use the Jones polynomials given in the knot tables to find a knot with eight crossings which cannot be deformed to its mirror image. Explain your reasoning.

(5) Find an oriented link which cannot be deformed to it's mirror image, yet has linking number zero. Explain how you know that your link cannot be deformed to its mirror image.

(6) Let L_+, L_-, and L_0 be three oriented link projections that are identical outside of a crossing where the three links differ as indicated below.



Follow the steps given below to prove the equation

$$q^{-1}V(L_+) - qV(L_-) + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})V(L_0) = 0$$

(a) First write equations for $\langle L_+ \rangle$ and $\langle L_- \rangle$ in terms of the variable A.
(b) Use the equations from step (a) to get an equation relating $\langle L_+ \rangle$, $\langle L_- \rangle$, and $\langle L_0 \rangle$.
(c) Use the equation from step (b) to get an equation relating $X(L_-)$, $X(L_+)$, and $X(L_0)$.
(d) Then use the substitution $q = A^{-4}$ in the equation from step (c) to get the required equation.

Lecture 11

Lectures 11 - 12 are based on *When Topology Meets Chemistry*, by Erica Flapan

1. Mirror Image Symmetry

We will now apply our knowledge of topology and knot theory to the study of molecular symmetries.

Molecules are often modeled as graphs in space, where vertices represent atoms or groups of atoms, and edges represent bonds. For example, the molecule L–alanine can be illustrated by the graph in Figure 1. The dark triangular segment in the figure indicates an edge which comes out of the plane of the paper towards you, the dashed segment indicates an edge which goes back behind the plane of the paper, and the ordinary line segments indicate edges which lie in the plane of the paper. In 3–dimensional space, the vertices of this graph lie at the corners of a regular tetrahedron.

L-alanine

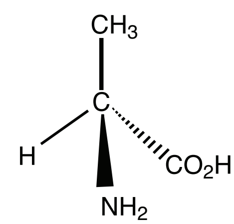


Figure 1. This molecule has the form of a tetrahedron

The 3-dimensional structure of a molecule determines many of its properties. For rigid molecules, like the one in Figure 1, this means its properties are determined by its geometry (i.e., by its bond angles and bond lengths). However, large molecules may be flexible, and some small molecules can even have pieces which rotate around specific bonds. For such molecules, topology can help us understand their structure.

For example, consider the molecular *Möbius ladder* illustrated in Figure 2. The graph is a 3-rung ladder where the ends have been joined with a half-twist. This molecule looks like a Möbius strip except it is made from a ladder rather than from a strip of paper. The sides of the ladder represent a polyether chain of 60 atoms which are all carbons and oxygens, and the rungs of the ladder represent carbon-carbon double bonds. The carbons are indicated by corners or vertices of the graph, and the oxygens are indicated by O's.

The molecular Möbius ladder was first synthesized in 1983 by David Walba. In order to synthesize a molecular Möbius ladder, Walba created a molecular ladder and then forced the ends of the ladder to join. These ladders did not all close up in the same way. The ends of some of the ladders joined together without a twist, the ends of some joined as Möbius ladders

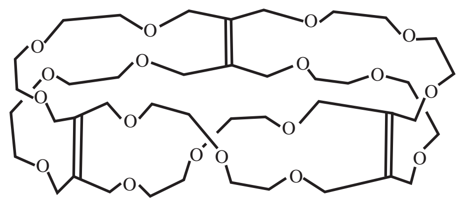


FIGURE 2. The molecular Möbius ladder

with a left-handed half-twist, and the ends of others joined as Möbius ladders with a right-handed half-twist. Figure 3 illustrates a ladder whose ends are joined without a twist.

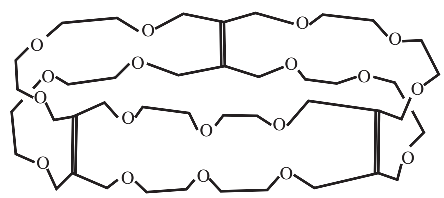


FIGURE 3. A molecular ladder in the form of a cylinder

Walba wanted to show that some of his ladders were Möbius ladders. The molecules were too small to see in a microscope, so Walba needed some other evidence to support his claim. He showed experimentally that some of his molecules were distinct from their mirror image. A cylindrical ladder, is the same as its mirror image, so not all of Walba's molecules were cylindrical ladders. Walba wanted to say that a molecular Möbius ladder is distinct from its mirror image. We illustrate the mirror image of Figure 2 in Figure 4. The two figures are identical except for the crossing in the front.

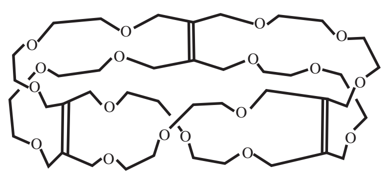


FIGURE 4. The mirror image of Figure 2

While Figure 4 looks different from Figure 2, it is not obvious that one structure could not somehow change itself into the other. In particular, since the Möbius ladder is flexible, perhaps there is is some way it can be deformed to its mirror image. In fact, if we do not distinguish the different types of atoms and bonds, then a Möbius ladder can be deformed to its mirror image. Figure 5 illustrates the Möbius ladder and its mirror image without specifying the atoms or distinguishing the different types of atoms or bonds. In the homework you will draw a deformation of one graph to the other.

FIGURE 5. A Möbius ladder and its mirror image without labeling the atoms or bonds

If we are considering a molecular structure we cannot ignore the difference between different types of atoms or different types of bonds. In 1983, the topologist, Jon Simon proved that if we distinguish between the different types of atoms, then the graph representing a molecular Möbius ladder cannot be deformed to its mirror image. It follows that a molecular Möbius ladder cannot change into its mirror form.

Knowing that the molecular Möbius ladder is different from its mirror image was an important step in recognizing the form of the molecule. In general, knowing whether a molecular structure is the same or different from its mirror image helps to predict its behavior. In order to refer to this property, we make the following definition.

Definition 1. *A molecule is said to be* **chemically achiral** *if it can change into its mirror image. Otherwise, it is said to be* **chemically chiral**.

Chemists say that a molecule is "chiral" or "achiral", rather than saying that it is "chemically chiral" or "chemically achiral." We insert the word "chemically" in our definition in order to distinguish this type of chirality from other types of chirality which we will introduce later. In particular, we want to make it clear that this is an experimental rather than a mathematical definition.

At first this definition may seem confusing because a molecule is chiral if it is NOT the same as its mirror image and it is achiral if it IS the same as its mirror image. Normally, we add the prefix "a" to a word to indicate that something is not the case. For example, consider the words political and apolitical. The definition makes more sense if you understand its etymology. The word *chiral* comes from the ancient Greek word $\chi\varepsilon\iota\rho$ which means hand. A left hand can never change into a right hand. Thus a hand is an example of a chiral object. So *chiral* really means "like a hand", and *achiral* means "not like a hand." In fact, we often think of the two forms of a chiral molecule as the left-handed form and the right-handed form.

Because biological organisms are chiral they react differently to the two forms of another chiral molecule, just as my left foot reacts differently to a right shoe and a left shoe. For example, the molecule *carvone* is chiral. One form smells like spearmint and the mirror form smells like caraway because the molecules in our olfactory systems are also chiral, and they are reacting to the carvone differently. It is believed that life cannot exist without chiral molecules. In fact, scientists are looking for chiral molecules on other planets as a sign of extraterrestrial life.

Pharmaceutical companies pay particular attention to chirality, because one form of a chiral medication is generally more effective while the mirror form generally has more side effects. Most medications are synthesized in a 50:50 mix, and separating the two forms is expensive though for some medications it is necessary. Also, sometimes when the two forms of a medication are separated they each have a separate use. For example, *Darvon* is a painkiller, while its mirror form *Novrad* is used as a cough medicine.

Thus it is important to know whether or not a molecule is chiral, and if it is chiral it may be useful to separate the two mirror forms. Chemists can determine experimentally whether or not a particular molecule is chemically chiral (as Walba did). But suppose a chemist is designing a new molecule, and wants to know before it is synthesized whether the molecule will be chiral. To do this, chemists generally use a geometric notion of chirality. Lord Kelvin, Professor of Natural Philosophy at the University of Glasgow, introduced this concept in 1884.

> "I call any geometrical figure, or group of points *chiral*, and say it has *chirality*, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself."

In other words, a structure is considered to be a rigid object and it is said to be chiral if it is different from its mirror image. For example, the molecule illustrated in Figure 1 is chiral because it's different from its mirror image. Modern organic chemistry textbooks often paraphrase Lord Kelvin's definition as follows.

Definition 2. *A* **chiral** *molecule is one that is not superimposable with its mirror image.*

This definition assumes that all molecules are rigid. We shall use the following definition which restates the above definition more mathematically.

Definition 3. *A structure is said to be* **geometrically achiral** *if, as a rigid object, it is identical with its mirror image. Otherwise, it is said to be* **geometrically chiral**.

Chemists use the word "chirality" for both geometric chirality and chemical chirality, because they consider these concepts to be interchangeable. However, these two concepts are not equivalent. For example, a right-handed glove is a geometrically chiral object, because as a rigid object it cannot be superimposed on a left-handed glove. Yet, if the glove is flexible (as a knitted or rubber glove is), then it can be turned inside out to get a left-handed glove. If the inside and outside of the glove are the same material and color, then in fact the glove is achiral even though it is not geometrically achiral.

In 1954, Mislow and Bolstadt synthesized a biphenyl derivative (illustrated in Figure 6) to demonstrate that chemical chirality and geometric chirality are not equivalent.

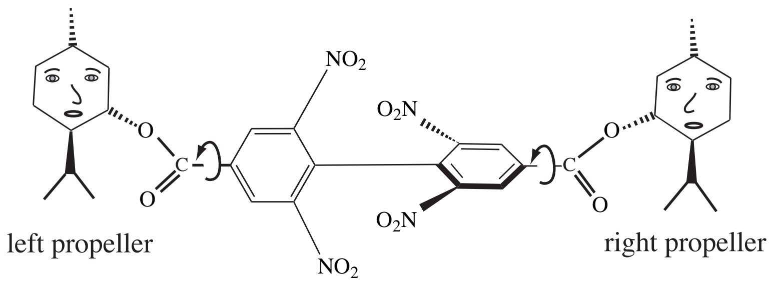


FIGURE 6. A molecule that is chemically achiral but geometrically chiral. (Note: the molecule does not actually have faces on it.)

In this figure, the hexagon to the right of the central bond is horizontal, while the hexagon to the left of the central bond is vertical. The vertical hexagons on either end of the structure are behind the plane of the paper, as indicated by the dashed lines attaching them to the rest of the graph. The image of the molecule in a vertical mirror will look the same except it will have a horizontal hexagon on the left of the central bond and a vertical hexagon to the right of the central bond. The mirror form is illustrated in Figure 7.

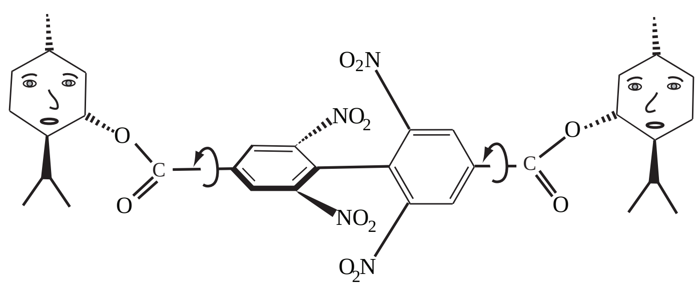


FIGURE 7. The mirror image of Figure 6

Figure 6 is rigid except that the two ends of the molecule rotate simultaneously as indicated by the arrows. We shall call these rotating pieces *propellers*. Of course the actual molecule does not have faces on it. The faces are drawn there to help illustrate the difference between the two propellers. The left propeller can be thought of as a person behind the plane of the paper facing you with her left hand forward, as indicated by the dashed bond connecting the hexagon face to the oxygen. The right propeller can be thought of as a person behind the plane of the paper facing you with her right hand forward. Because of the rigidity of the propellers, the notion of "forward" is unambiguous. The faces help us remember that forward is the direction the hands are pointing in. As rigid structures, a right propeller can never become a left propeller (as we can see by imagining the propellers as people).

We see that the molecule in Figure 6 is chemically achiral as follows. Rotate the whole structure by $90°$ about a horizontal axis. After the rotation, the hexagon on the left is horizontal and the hexagon on the right is vertical, and the hexagons in the propellers are horizontal. Since the propellers rotate, we let them rotate back to their original vertical positions. In this way we obtain the structure in Figure 7. Since this motion can occur on a chemical level, we conclude that the structure is chemically achiral.

We see as follows that the molecule in Figure 6 is geometrically chiral. Observe that in Figure 6 the hexagon in the left propeller is parallel to the adjacent hexagon. In the mirror form the hexagon in the left propeller is perpendicular to the adjacent hexagon. Since, as a rigid object, there is no way for a left propeller to become a right propeller, there is no way to rigidly move this structure so as to superimpose it on its mirror image.

This phenomenon is occurring because the molecule is rigid everywhere except where the propellers rotate. If the molecule were completely rigid it would be chemically chiral. On the other hand, if it were completely flexible, it could lie in a plane and hence would become geometrically achiral. Below we define a term introduced by the chemist Van Gulick to describe examples of this type.

Definition 4. *A molecular structure is said to be a* **Euclidean rubber glove** *if it is chemically achiral, but it is chemically impossible for it to attain a position which can be rigidly superimposed on its mirror image.*

In order to understand this concept, we first consider a real rubber glove. As we observed above, a right-handed rubber glove becomes a left-handed rubber glove when it is turned it inside out. However, due to the physical limitations of rubber, the glove cannot attain a position which can be rigidly superimposed on its mirror image. Just as with the above molecule, if the glove were completely flexible, it could be flattened into the plane so that it would be its own mirror image, and hence it would be geometrically achiral. Van Gulick used the word "Euclidean" in this definition to indicate that physical or geometric constraints are what prevents the structure from attaining a geometrically achiral position.

The chemist David Walba asked if there could ever be a chemically achiral molecule that could not be deformed to a geometrically achiral position, even ignoring physical or geometric constraints. In particular, Walba proposed the following definition, and asked if there exists a molecule which satisfies this definition.

Definition 5. *A molecular structure is said to be a* **topological rubber glove** *if it is chemically achiral, but even if it were completely flexible it could not be deformed to a position which could be rigidly superimposed on its mirror image.*

At first glance it seems impossible for a structure to be a topological rubber glove. If a structure can change into its mirror image and we ignore all physical and geometric constraints, shouldn't we be able to deform it to a symmetric position, which can be rigidly superimposed on its mirror image? But, as it turns out this is not always the case.

Figure 8 illustrates the first molecule which was shown to be a topological rubber glove. This molecule was synthesized by Chambron, Sauvage, and Mislow in 1997. A key feature of this molecule is that the pair of adjoining hexagons at the top of the molecule can rotate about the bonds that connect them to the rest of the molecule (as indicated by the arrows). Note that the H_3C and the CH_3 in the mirror image are the same group of atoms. They are written as they are to indicate that the C is attached to the adjacent hexagon in both molecules.

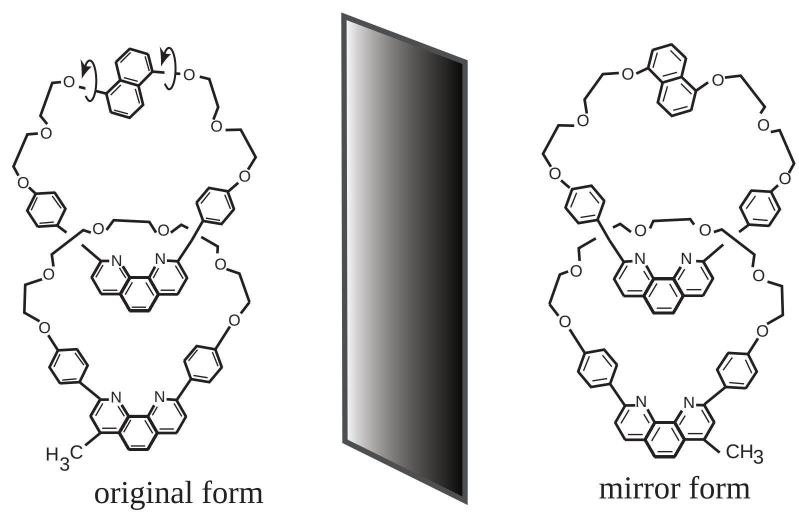


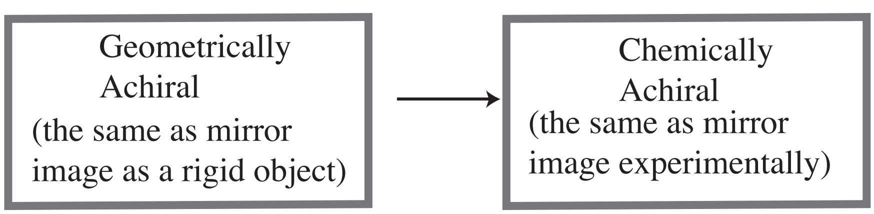
Figure 8. A topological rubber glove

If we turn the lower ring of this molecule over and rotate the pair of adjoining hexagons at the top, the molecule in Figure 8 becomes its mirror image. Since both of these motions can occur on a chemical level, the molecule is chemically achiral. On the other hand, it can be shown that even if the molecule in Figure 8 were completely flexible it could not be deformed to a position which could be rigidly superimposed on its mirror image. We give an intuitive idea of the proof here. The H_3C in the original form gives the bottom ring an orientation. If the molecule were rigid, then the hexagons at the top cannot rotate, and hence they give the top ring an orientation. Thus we can model this molecule as an oriented Hopf link (see Figure 9).

FIGURE 9. An oriented Hopf link cannot be deformed to its mirror image.

Since one form of the oriented Hopf link has linking number 1 and the mirror form has linking number -1, there is no deformation of an oriented Hopf link to its mirror image. It follows that there is no deformation from the molecule in Figure 8 to a position which can be rigidly superimposed on its mirror image. Hence the molecule in Figure 8 is a topological rubber glove.

We summarize the relationship between geometric and chemical chirality as follows. If a structure can be rigidly superimposed on its mirror image, then it is physically the same as its mirror image. Hence it must be chemically the same as its mirror image. Thus geometric achirality implies chemical achirality. It may be easier to remember with a picture.



Taking the contrapositive of this statement, we see that chemical chirality implies geometric chirality. On the other hand, Euclidean and topological rubber gloves are examples of molecules which are geometrically chiral but not chemically chiral. Hence geometric chirality does not imply chemical chirality.

When we defined geometric chirality, we were treating all molecules as if they were completely rigid. As we have seen some molecules are rigid, some are flexible, and some have pieces can rotate around certain bonds. A mathematical characterization of all molecular symmetry groups is impossible because the level of rigidity depends on the chemical properties of each individual molecule. Now we will take the opposite point of view from the rigid one with started with and treat all molecules as if they were completely flexible.

Definition 6. *A structure is said to be* **topologically achiral** *if it can be deformed to its mirror image (assuming complete flexibility). Otherwise, it is said to be* **topologically chiral**.

Since any molecular motion is a deformation, chemical achirality implies topological achirality. Thus topological chirality implies chemical chirality. However, topological achirality does not imply chemical achirality, as Figure 10 illustrates. Since this molecule is rigid, it cannot be superimposed on its mirror image. On the other hand, if the graph were flexible, we could interchange the positions of CO_2H and NH_2. Thus the structure is topologically achiral.

L-alanine

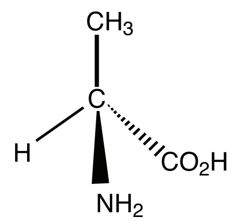
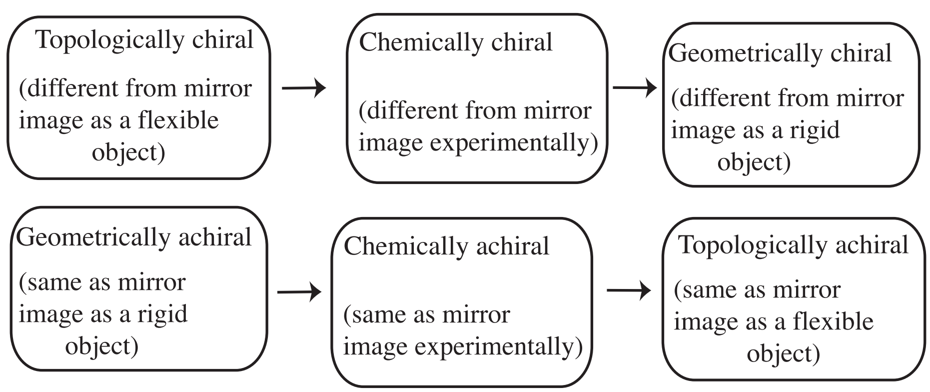


FIGURE 10. This molecule is topologically achiral but chemically chiral

So geometry enables us to prove chemical achirality, and topology enables us to prove chemical chirality. Again perhaps a picture is worth a thousand words.

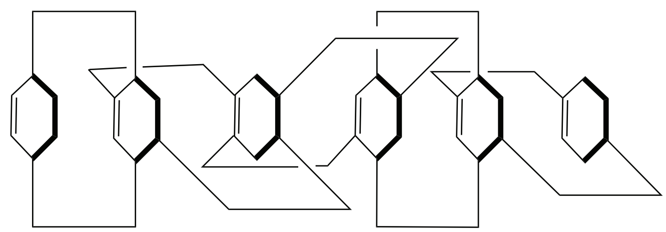


Furthermore, a molecule whose chirality comes from its rigidity can be forced to change into its mirror image by heating it. However, if a molecule is topologically chiral, then bonds would have to break for it to change into its mirror image. Thus it will not change into its mirror form even when heated. So topological chirality is a more enduring type of chirality than geometric chirality. All of this is to say that topological chirality is a useful concept for chemists.

Homework

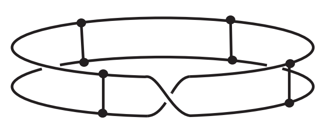
(1) Explain why geometric chirality does not imply chemical chirality, and chemical chirality does not imply topological chirality.

(2) Prove that the molecule illustrated below can be deformed into a plane if it is completely flexible. Is it geometrically chiral? Is it topologically chiral? Given that this molecule is rigid, is it chemically chiral?



(3) (hard) Prove that if we don't distinguish between rungs and sides then the embedded graph of a 3-rung molecular Möbius ladder can be deformed to its mirror image.

(4) Draw a step-by-step deformation taking a Möbius ladder with four rungs (illustrated below) to a position where the circle representing the sides of the ladder lies in a plane.



(5) Consider a closed ladder with three half twists, instead of just one. Prove that there is no deformation that takes it to its mirror image taking sides to sides and rungs to rungs.

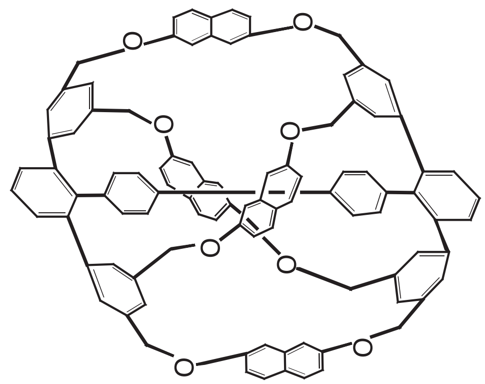
(6) Is the molecule illustrated in Figure 6 a topological rubber glove? Explain your reasoning.

(7) Prove that if a molecule is a topological rubber glove then it is a Euclidean rubber glove, but not the converse.

(8) We define an object to be a *Euclidean rubber glove* if it can be deformed to its mirror image, but it cannot be deformed to a position which could be rigidly superimposed on its mirror image. Explain why a real rubber glove a Euclidean rubber glove. Give another example of a familiar object which is a Euclidean rubber glove, and explain why.

(9) We define an object to be a *topological rubber glove* if it can be deformed to its mirror image, but even if its completely flexible it cannot be deformed to a position which could be rigidly superimposed on its mirror image. Is a trefoil knot a topological rubber glove? What about a figure eight knot? What about a Hopf link?

(10) (hard) Prove that the molecular graph of Kuratowski cyclophane (which is illustrated in below) can be deformed to a geometrically achiral position.



Lecture 12

1. Techniques to Prove Topological Chirality

Geometric chirality is not hard to determine by examining a rigid structure from all different angles to see whether or not it can be superimposed on its mirror image. Proving that a structure is topologically chiral is more difficult because we cannot consider all possible deformations of the structure. However, a number of useful techniques have been developed to prove topological chirality.

Technique 1: Use a knot or link invariant.

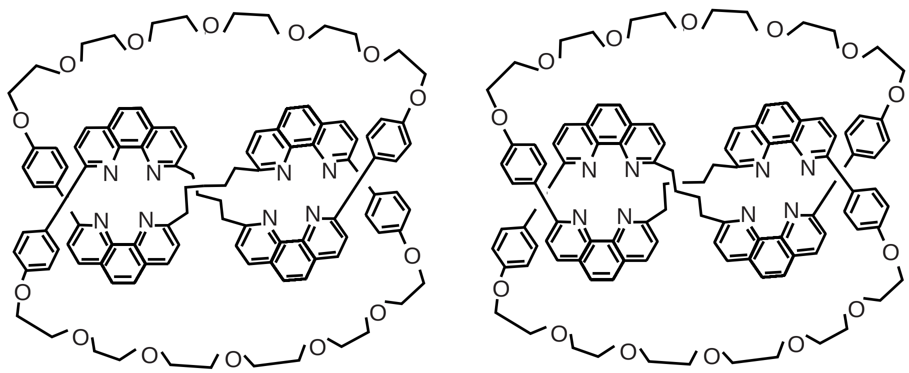


Figure 1. A molecular trefoil knot and its mirror image

You showed in the exercises on knot polynomials that the X-polynomial of a knot is the same as the X-polynomial of the mirror image of the knot except that the signs of the exponents have been reversed. So if we compute the X-polynomial of a knot and observe that it is not symmetric with respect to the signs of its exponents, then we immediately know that the knot is topologically chiral. This is also true for the Jones polynomial. We can conclude that the molecule in Figure 1 is topologically chiral by simply looking in a knot table and finding that its Jones polynomial is $q + q^3 - q^4$, which has only positive exponents, and hence does not have exponents which are symmetric with respect to sign.

One note of caution, just because a knot has the same Jones polynomial as its mirror image does not mean that the knot is topologically achiral. Thus the Jones polynomial can be used to show topological chirality and but not to show topological achirality. Also, knot or link invariants cannot be used to prove topological chirality if a molecule contains no knots or links. For example, we can't use such invariants to prove the molecular Möbius ladder (illustrated in Figure 2) is topologically chiral.

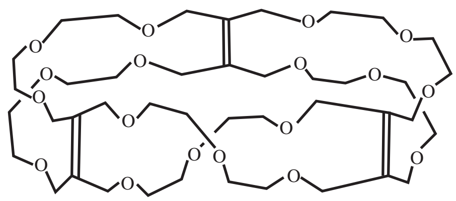


FIGURE 2. A molecular Möbius ladder

Technique 2: Use a 2-fold branched cover

While 2-fold branched covers have been around for a long time, the idea of using them to prove that a molecule is topological chiral was first introduced by Jon Simon in his analysis of the molecular Möbius ladder. Since the Möbius ladder contains no knots or links, a knot invariant will not help us prove that it's topologically chiral. Simon came up with the idea of using 2-fold branched covers. We won't define branched covers because the definition is too technical, but we will give some intuition about how to build the branched cover and how this method was used.

Rather than drawing a molecular Möbius ladder with all the atoms labeled, we represent it symbolically as a colored graph where the rungs are black and the sides are grey (see Figure 3). Since it would not make sense chemically to deform the graph so that an oxygen goes to a carbon, we make the rule that edges of a given color cannot be deformed to edges of a different color. We draw the rungs as three different types of segments, not because they are different, just to help the reader see what is going on later in our argument.

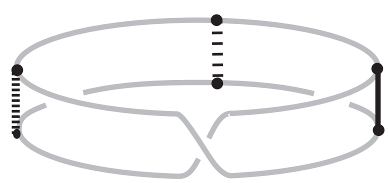


FIGURE 3. A colored Möbius ladder

We assume this graph is completely flexible, and deform the sides of the ladder to be a planar circle. After this deformation, the graph looks like the graph on the right in Figure 4. (You should check that the graph can be deformed to look like this.) We have numbered the rungs so that rung 1 is at the bottom, rung 2 is in the middle, and rung 3 is at the top.

Creating the 2-fold branched cover of our graph involves gluing two copies of the graph together along the grey circle. In this way, we obtain the structure illustrated in Figure 5, which has only one grey circle, and three additional rungs, $1'$, $2'$ and $3'$, directly under the original three, such that rungs $1'$, $2'$ and $3'$ are in the same vertical order as rungs 1, 2, and 3, and

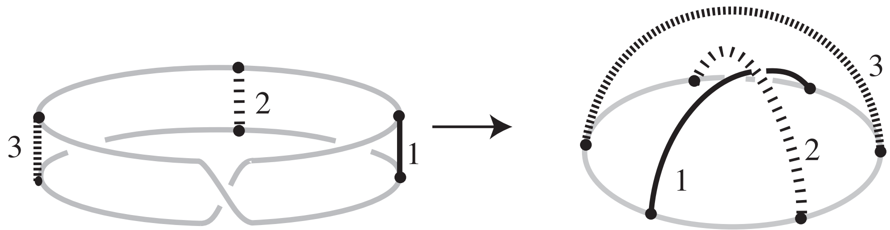


Figure 4. A colored Möbius ladder with planar sides and numbered rungs

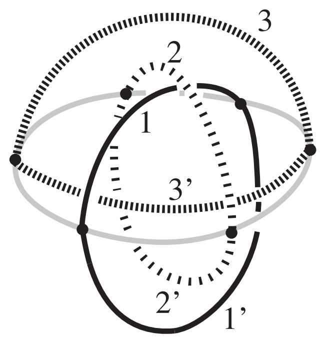


Figure 5. The 2-fold branched cover of the Möbius ladder

rung $1'$ has the same endpoints as rung 1, rung $2'$ has the same endpoints as rung 2, and rung $3'$ has the same endpoints as rung 3.

If we erase the grey circle from Figure 5 we obtain a link with three components, which has the property that each component of the link is linked to each of the other components (see Figure 6). Simon proved that this link is topologically chiral by using the linking number. This requires considering cases according to what orientation we put on each component. Now the topological chirality of this link implies that the colored Möbius ladder must be topologically chiral. Hence the molecular Möbius ladder is also topologically chiral.

Technique 3: Find a unique Möbius ladder in the molecule.

This method was developed by the chemist Kurt Mislow to prove that the molecule *triple layered naphthalenophane* (illustrated in Figure 7) was topologically chiral.

We illustrate triple layered naphthalenophane in Figure 7. A naphthalene consists of a pair of hexagons, and is the primary ingredient in mothballs. We see that this molecule does indeed have three layers of naphthalene.

A key observation that we use is that the graph of triple layered naphthalenophane contains a unique circuit that is longer than any other circuit

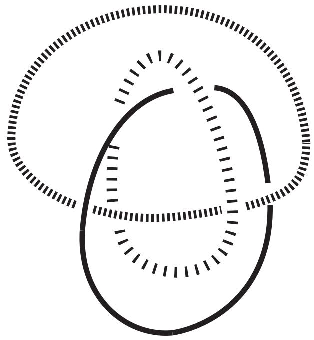


FIGURE 6. If we erase the grey circle in Figure 5 then we have this link.

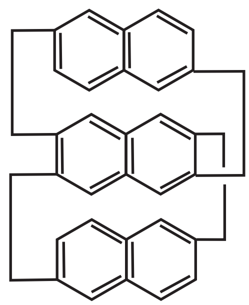


FIGURE 7. Triple layered naphthalenophane

in the graph. In Figure 8, we have illustrated this longest circuit. You should check that this circuit is indeed longer than any other circuit in the graph.

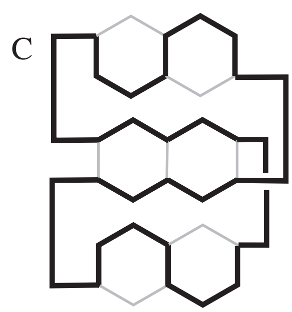


FIGURE 8. The unique longest circuit in triple layered naphthalenophane

The three vertical edges in the center of the graph are the only edges which have both endpoints on this unique circuit. In Figure 9, we illustrate the unique circuit together with these three edges (without the additional edges at the top and bottom of the figure).

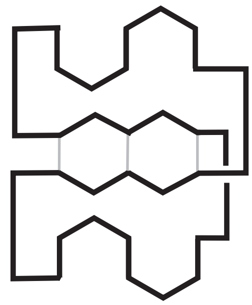


FIGURE 9. The longest circuit together with the three edges whose vertices are on the circuit

Suppose, for the sake of contradiction, that the graph of triple layered naphthalenophane can be deformed to its mirror image. Such a deformation would take the colored graph in Figure 9 to its mirror image. However, Figure 9 can be deformed to a colored Möbius ladder, which we know is topologically chiral. Hence such a deformation is not possible. Therefore, triple layered naphthalenophane must be topologically chiral.

Technique 4: A combinatorial approach

The following method focuses on the structure of the graph, rather than the way the graph is embedded in 3-dimensional space. We begin with several definitions.

Definition 1. *An* **automorphism** *of a graph is a permutation of the vertices which takes adjacent vertices to adjacent vertices. For molecular graphs an automorphism is also required to take atoms of a given type to atoms of the same type.*

For example, consider the Möbius ladder illustrated in Figure 10. We have numbered the vertices to make it easier to describe a particular permutation of the vertices. The permutation which interchanges vertices 2 and 6 and interchanges vertices 3 and 5 is an automorphism. On the other hand, the permutation which just interchanges vertices 2 and 5 is not an automorphism because vertex 2 is adjacent to vertex 3, while vertex 5 is not adjacent to vertex 3.

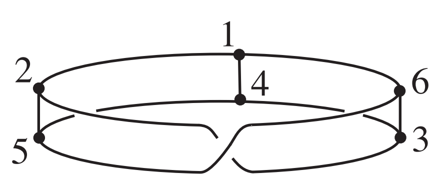


FIGURE 10. The permuation which interchanges vertices 2 and 6 and vertices 3 and 5 is an automorphism

The *valence* of a vertex is the number of edges that contain that vertex. For example, in a Möbius ladder all of the vertices have valence 3. The *distance* between two vertices in a graph is the fewest number of edges contained in a path from one to the other.

You will prove the following properties of automorphisms in the homework.

1) Any automorphism of a graph takes vertices of a given valence to vertices of the same valence.
2) Any automorphism of a graph takes a pair of vertices which are a certain distance apart to a pair of vertices which are the same distance apart.

Definition 2. *The* **order** *of an automorphism is the smallest number n such that doing the automorphism n times takes each vertex back to its original position.*

For example the automorphism of the Möbius ladder which interchanges vertices 2 and 6 and vertices 3 and 5 has order two. The identity automorphism (also known as the *trivial automorphisms*) that doesn't move any vertex, has order one.

The graphs K_5 and $K_{3,3}$ (see Figure 11) are special because any graph which contains one of these graphs cannot be embedded in a plane. If a graph is embedded (i.e., situated) in a plane then it is its own mirror image, hence it is topologically achiral.

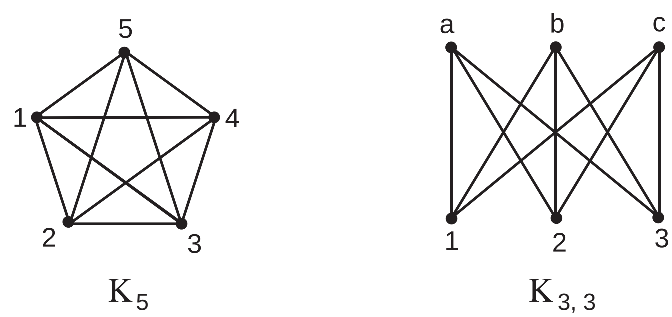


FIGURE 11. Drawings of the abstract graphs K_5 and $K_{3,3}$. Note the edges do not intersect outside of the vertices.

The following theorem allows us to prove that many graphs are topologically chiral. This result has nothing to do with the way the graph is embedded in 3-dimensional space. So if it applies to one embedding of the graph then it applies to all embeddings of that graph.

Theorem. *If a graph contains either of the graphs K_5 or $K_{3,3}$ and has no order two automorphisms, then any embedding of the graph in $\mathbb{R}^3$ is topologically chiral.*

Observe that this result has nothing to do with the particular embedding of the graph in $\mathbb{R}^3$. So once we check the conditions, then even if we change the embedding (i.e., change the extrinsic topology of the graph in $\mathbb{R}^3$) the graph will still be topologically chiral. This observation motivates us to make the following definition.

Definition 3. *A graph is said to be* **intrinsically chiral** *if every embedding of it in* $\mathbb{R}^3$ *is topologically chiral.*

That is, a graph is intrinsically chiral if its topological chirality is an intrinsic property of the graph and doesn't depend on the extrinsic topology of the graph in $\mathbb{R}^3$. Using this terminology, we can restate the above theorem as follows.

Theorem. *If a graph contains either of the graphs* K_5 *or* $K_{3,3}$ *and has no order two automorphism, then it is intrinsically chiral.*

To see how to apply this theorem we consider the molecule ferrocenophane, which is illustrated in Figure 12. All of the atoms of this molecule are carbons except for a single iron (indicated with an Fe) and a single oxygen (indicated with an O).

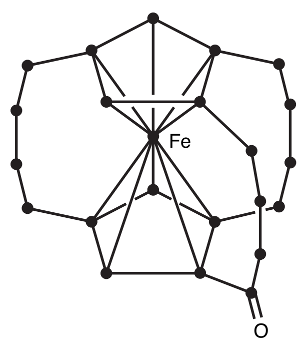


FIGURE 12. The molecule ferrocenophane

While it may not be obvious at first glance, ferrocenophane contains the complete graph K_5. The graph K_5 consists of five vertices together with edges between every pair of vertices. We illustrate the K_5 contained in ferrocenophane in Figure 13.

In order to prove that ferrocenophane has no order two automorphisms we argue as follows. Suppose that Φ is an automorphism of ferrocenophane. Since Φ has to take atoms of a given type to atoms of the same type, it must take the single oxygen atom to itself. Now since Φ preserves adjacency, Φ takes the vertex which is adjacent to the oxygen atom to itself. There are two vertices adjacent to this vertex, however one has valence two and the other has valence four. Thus Φ must take each one to itself. We can continue going through the vertices, arguing in this way, to conclude that every vertex of ferrocenophane must be fixed by the automorphism Φ. But this means that

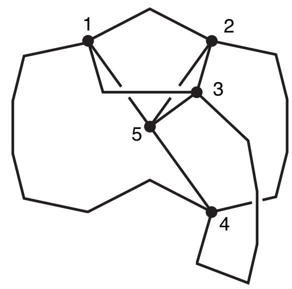


FIGURE 13. Ferrocenophane contains the graph K_5

Φ has order one. Hence ferrocenophane has no automorphism of order two. Now it follows from the above theorem that ferrocenophane is intrinsically chiral (which means that not only is this molecule topologically chiral, but any molecule which has the same abstract graph is topologically chiral as well).

Homework

(1) Prove that the link in Figure 6 is topologically chiral.

(2) Prove that the circuit C in Figure 8 is longer than any other circuit in the graph.

(3) Give a step-by-step deformation from Figure 9 to a colored Möbius ladder.

(4) Consider a graph like triple layered naphthalenophane, but with a row of four hexagons on each of the three layers instead of just two. Prove that this graph is topologically chiral.

(5) Prove that the graph $K_{3,3}$ is not intrinsically chiral.

(6) Prove that the Simmons-Paquette molecule is intrinsically chiral.

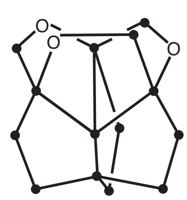


FIGURE 14. The Simmons-Paquette molecule

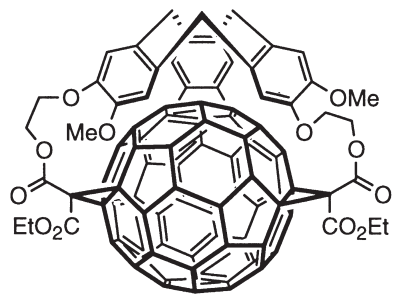
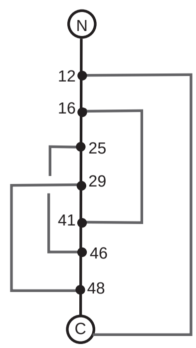


FIGURE 15. Prove that this graph is non-planar

(7) Prove that the graph in Figure 15 is non-planar, and contains a special circle such that any automorphism of it takes that circle to itself.

(8) Prove that the molecule in Figure 15 is intrinsically chiral.

(9) Prove that the protein illustrated below is topologically chiral.



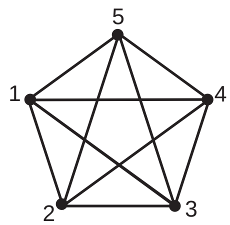
(10) Prove that any automorphism of a graph takes vertices of a given valence to vertices of the same valence.

(11) Prove that any automorphism of a graph takes a pair of vertices which are a given distance apart to a pair of vertices which are the same distance apart.

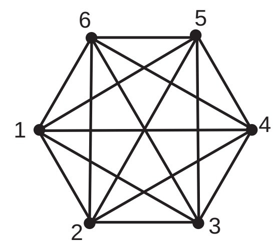
(12) Prove that a molecular Möbius ladder has an automorphism of order 6.

(13) Let C denote a circle with n vertices. Prove that the order of any automorphism of C is either 2 or divides n.

(14) Find topogically achiral embeddings of the complete graphs K_5 and K_6 illustrated below. Note there are no intersections between the edges. We just draw them that way because we don't want to specify an embedding of the graphs in space.

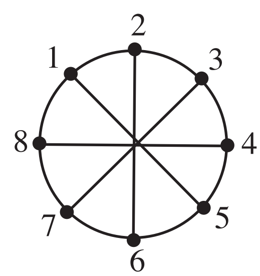


K_5



K_6

(15) As an abstract graph, a Möbius ladder with four rungs is a circle with four additional edges connecting opposite points on the circle. Thus if you travel around the circle you pass through the first endpoint of each rung and then you pass through the second endpoint of each rung in the same order. Below is an illustration of the abstract graph of a Möbius ladder with four rungs. Note the edges do not actually intersect in the center of the picture. Draw a topologically achiral embedding of this graph.



(16) Show that the graph illustrated below contains the abstract graph of a Möbius ladder with four rungs.

(17) Prove that the graph illustrated below is topologically achiral.