Problem Set 3: Suln'hfig

Today's title is a rejected name of a Sesame Street character. The character's real name is much harder to spell. Wait, no, it's actually the Klingon word for killing someone while shuffling.

Opener

With an even number of cards, there is a different way to do a "perfect shuffle" by starting from the bottom half instead of the top. For example,

$$123456 \Rightarrow 415263$$

What changes? Determine the number of shuffles needed for different deck sizes.

Important Stuff

1. What's the difference between *mod 7* and *base 7*? Write a brief explanation (with a numerical example) that a middle-school student could understand.

2. Using the shuffle from *today's opener*, describe the path taken by the top card when you repeatedly shuffle an eight-card deck.

Except for holidays and weekends, we seem to be shuffling at least once in each 24-hour period.

3. These are the *entire tables* for addition and multiplication in mod 7. FINISH THEM!!!

MODALITY! Sub-zero equals six. But only in mod 7. Get it? Whatever.

+	0	1	2	3	4	5	6
0							
1							
2				5			
3							
4							
5			0				
6							5

×	0	1	2	3	4	5	6
0							
1							
2				6			
3	0						
4							
5			3				
6							1

4. Use the tables you built to find all solutions to each equation *in mod 7*. Some equations may have more than one solution, while others may have none.

   a. $5 + a = 4$
   b. $4 \cdot b = 3$
   c. $(5 \cdot c) + 6 = 1$
   d. $d^2 - 4 = 0$

A Higgs boson walks into a church. "What are you doing here?" asks the priest. "You can't have mass without me!" replies the Higgs boson.

5. **a.** Build addition and multiplication tables for mod 10.
b. Solve the four equations from Problem 4 in mod 10.

6. **a.** Are there negative numbers in mod 7? Does any number behave like -1?

Sub-zero says hi.

b. Are there perfect squares in mod 7? How many?
c. Are there powers of 2 in mod 7? How many?

The number 1 is a power of any positive integer b, since $b^0 = 1$.

7. **a.** Gabriel is calculating the powers of 3 in mod 100. Compute the next three entries in the sequence.

$$1, 3, 9, 27, 81, 43, 29 \dots$$

DOES NOT COMPUTE . . . OH WAIT IT TOTALLY DOES. Figure out how to do this without calculating $3^7 = 2187$.

b. Compute the sequence of powers of 3 in mod 3. Uhh.
c. Gail declares that mod 3 wasn't very interesting, and demands that you compute the sequence in mod 7.

Gail = Gabriel + US - rebus!

Neat Stuff

8. Jason handed us a cute blue Post-It that said:

It was both very much like and very much unlike a Smurf.

$$10^2 + 11^2 + 110^2 = 111^2.$$

a. Surely the numbers 10, 11, 110, and 111 in the note are in base 2. Check to see if the statement is true in base 2.
b. Hey wait, maybe those numbers are in base 3. Check to see if the statement is true in base 3.
c. Oh, hm, maybe it was in base 4.
d. Sorry, it was actually in base n. What!

9. **a.** Find all the powers of 3 in mod 9. Oh, that was exciting.
b. Find all the powers of 2 in mod 9.

10. **a.** If $n > 2$, is it possible for *every number* to be a power of 2 in mod n?
b. If $n > 2$, is it possible for *every number except 0* to be a power of 2 in mod n?

Alright, part (b) kind of gives away the answer to part (a) here.

11. Euler conjectured that it takes at least k kth powers to add up to another one. For example, $3^2 + 4^2 = 5^2$ but you need three cubes to add up to another cube. In the 1960s this was finally disproven:

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Without a calculator, and hopefully without multiplying it all out, find the value of n.

12. In the opener, we said you could only do this other shuffle with an even number of cards. We lied. Figure out how to do today's shuffle with an odd number of cards. What do you notice?

We finally found the Higgs boson, so now we can redirect all efforts on finding that darn Waldo.

13. Arielle, Becky, and Chelsea were standing in Tuesday's ridiculous lunch line and had an idea. If any two neighbors switch places, it would create a different arrangement... like

$$\text{ABC} \Rightarrow \text{ACB}$$

They decided to make a big graph of all six ways they could be arranged, and all the connections that could lead from one way to another. Your turn!

Note that only *neighbors* may switch places. Arielle and Chelsea can trade places, but not as the first move.

14. Darren gets in the back of the line behind Arielle, Becky, and Chelsea, and they realize they're going to be stuck making a much larger graph. Good luck!

This graph will contain lots of little copies of the last graph! Neat.

15. If a number can be represented as a repeating decimal in base 10, does it have to be a repeating decimal in every other base? If yes, explain why. If no, are there any particular bases in which it *must* be a repeating decimal?

16. Today is 7/5/12, and $7 + 5 = 12$. Oh snap!

- **a.** How many more times this century will there be a day like this? By *this* we mean the next one is August 4, 2012.
- **b.** How many times will Buck Rogers in the 25th Century see a day like this? You may assume that Buck Rogers arrives on January 1, 2401 and remains alive through the entire century.
- **c.** How can the second answer help you check the first?

And the last one is . . . later than the others?

Wow, that is a really cool answer!

17. Predict the length of the base-10 repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

Tough Stuff

18. Predict the length of the *base-2* repeating decimal expansion of $\frac{1}{107}$, then see if you were right.

19. For even n, the maximum number of perfect shuffles needed to restore a deck with n cards to its original state appears to be $n - 2$. Find a rule that tells you when an n-card deck will have the maximum number of necessary perfect shuffles.

A Higgs boson walks into a bar. "Want a drink?" asks the barman. The Higgs boson doesn't reply, because it's a Higgs boson, not a person.

20. *It's p-adic number time!* Every 2-adic positive integer looks like it normally does in base 2, except it has an infinite string of zeros to the left. For example

$$7 = ...000000000000000111.$$

Where he at, where he at . . . p-adic numbers, p-adic numbers, p-adic numbers with a baseball bat!

a. Verify that $7 + 4 = 11$ using 2-adic arithmetic.
b. What about subtraction? Try $4 - 3$, and then try $3 - 4$.
c. Compute the sum

$$...111111111111111. + ...0000000000001.$$

Oh *man* are you going to have to do a lot of borrowing!

d. Compute this sum using 2-adic arithmetic:

$$1 + 2 + 4 + 8 + 16 + \cdots$$

21. Judy's favorite number is the golden ratio $\phi = (1+\sqrt{5})/2$ and the other Judy loves to do arithmetic in base ϕ. The only problem is that even if the only allowable digits in base ϕ are 0 and 1, not every number has a unique representation. Prove that every number has a unique base-ϕ representation if two consecutive 1s are disallowed.

22. **a.** Find all solutions to $x^2 - 6x + 8 = 0$ in mod 105 without use of any technology. There's a lot of them.
b. Find all solutions to $x^2 - 6x + 8 = 0$ in mod 1155 without use of any technology.

Tell you what, we'll give you two solutions: $x = 2$ and $x = 4$. See, now it's not nearly as tough.