Problem Set 4: Shuf'ling Shuf'ling

Opener

Write down all the powers of 2 in mod 17. Write down all the powers of 2 in mod 13.

Perform Thursday-style shuffles on a 16-card deck, tracking the position of the first card. Do it again for a 12-card deck.

On Monday, perfect shuffles were like the McDLT: the top card stayed on top, and the bottom card stayed on the bottom. Thursday's perfect shuffles were more like the McRib, since they made everything move. Wow, what a horrible analogy.

Important Stuff

1. Complete this table.

n	Powers of 2 in mod n	Cycle Length
7	$1, 2, 4, 1, 2, 4, 1, \ldots$	3
9		
11		
13		
15		
17		
19		
21		
23		
25		
27		

2. If you perform Thursday-style shuffles on a 24-card deck, what positions will the top card take?

MORE shuffling? Oh man, in all possible time spans, we're shuffling.

3. **a.** Kathryn has a deck of 12 cards. Write out the order of her cards after a few of the Monday-type shuffles.

1	2	3	4	5	6	7	8	9	10	11	12

b. Kathy has a deck of 12 cards. Write out the order of her cards after a few of the Monday-type shuffles.

0	1	2	3	4	5	6	7	8	9	10	11

c. Kathi has a deck of 10 cards. Write out the order of her cards after a few of the Thursday-type shuffles.

1	2	3	4	5	6	7	8	9	10

d. Why might you want to number the cards starting from 0 or 1 for a particular type of shuffle? Does anything change with an odd number of cards?

Girl look at those numbers. They work out!

4. What are the powers of 2 in mod 51?

5. Explain why, in the 52-card deck we saw on Monday, the second card in the deck returns to its original position in 8 shuffles.

It's times like you would think a Shufflebot would be useful. Sadly, Shufflebot is only programmed to dance and to apologize.

Neat Stuff

6. Perform Thursday-style shuffles on a 20-card deck and track where each card goes. Complete this table.

Card No.	Positions	Cycle Length	Card No.	Positions	Cycle Length
1			11		
2			12		
3			13		
4	$4, 8, 16, 11, 1, 2, 4, \ldots$	6	14		
5			15		
6			16	$16, 11, 1, 2, 4, 8, 16, \ldots$	6
7			17		
8			18		
9			19		
10			20		

7. All 52 cards from the deck we saw on Monday have a cycle. List all 52 cycles. Maybe there is some way to do it without listing the entire cycle for every single card?

8. Really, aren't you sick of calling these Monday and Thursday shuffles? What should we call them? Best names win!

The worst possible names for these shuffles are SkyBlu and Redfoo.

9. The last problem set asked you to find the perfect squares and powers of 2 in mod 7.

a. You can build a multiplication table to find all the perfect squares in mod 15, but there might be other ways. How many perfect squares are there in mod 15?

b. Refer to Problem 1. How many powers of 2 are there in mod 15?

Compare to the result from Problem 6 from Day 3.

10. Our favorite repeater, $\frac{1}{7}$, can be written as a "decimal" in each base between base 2 and base 10. Find each expansion and see if they have anything in common.

11. Corey, Debbie, and Erik are waiting in line, wondering if they can get to any arrangement through these two rules:

- The person in the back of the group may jump to the front: XYZ $\Rightarrow$ ZXY
- The two people at the front of the group may swap places: XYZ $\Rightarrow$ YXZ

Kan all six possible arrangements be made? Make a graph illustrating the options.

Swaps, swaps, swaps! Swaps swaps swaps!

Sorry for party typo. Or *is* it? Perhaps it completes a phrase. Fun fact: LMFAO won the Kids' Choice award for Favorite Music Group, forcing them to declare their name stood for Loving My Friends And Others.

12. Fred joins the back of the group. Under the same rules, decide whether or not all 24 possible arrangements can be made, and make a graph illustrating the options.

13. So $2^8 = 1$ in mod 51. All this actually proves is that the *first* moving card returns to its original position after 8 shuffles. Complete the proof by showing that every other card also returns to its original position after 8 shuffles.

14. Sometimes while shuffling, the deck completely flips. When this happens, all cards appear in reverse order (except for the end cards when using a Monday-style shuffle). Some people observed that when this happens, that was a halfway point to the shuffling. Is this true? Explain why or why not.

Flipping your deck while shuffling sounds like one of the greatest breakdancing moves ever. Breakdancing hasn't been the same since they cancelled production on Breakin' 3: The Boogaloo Kid.

15. When using Thursday-style shuffles, for what deck sizes does the deck completely flip?

16. Monday's 52-card shuffle didn't have a flipped deck at 4 shuffles, because we would have noticed that. But does anything interesting happen at the 4th shuffle? Look carefully and compare the deck after 4 shuffles to the original deck. Can you explain why this happens?

The observation here may be easier with two decks of cards; one to shuffle, and one to leave in the original setup.

17. Find a mathematical equation that is true in mod 2 and mod 3, but not true in general.

18. Find a mathematical equation that is true in mod 2, mod 3, mod 4, and mod 5, but not true in general.

Psst: you can skip mod 2. Why?

19. Investigate any connection between the number of powers of 2 in prime mods and the number of powers of 2 in composite mods. Look for an explanation or proof of what you find.

20. *It's p-adic number time!* In 3-adic numbers, non-negative integers are written in base 3 with leading zeros:

$$16 = ...00000000000121.$$

a. Try $16 - 9$. Hey, that wasn't so bad!
b. Try $16 - 17$. Oh dear.
c. What is the value of $1 + 3 + 9 + 27 + 81 + \cdots$?

Aww yeah! The p-adic numbers make as much sense as most LMFAO videos.

Tough Stuff

21. The length of the repeating decimal for $\frac{1}{2}$ in base p, where p is prime, is sometimes even and sometimes odd. When? Find a rule and perhaps a proof even?

The first person to find and prove this will receive a champagne shower! Offer expires 7/5/2012.

22. For what primes p is there an even length of the repeating decimal for $\frac{1}{5}$ in base p?

23. For what primes p is there an even length of the repeating decimal for $\frac{1}{10}$ in base p?

24. Determine and prove the Pythagorean Theorem for p-adic numbers, or decide that this problem is completely bogus and there is no such thing.

It's mathy and you know it.