Problem Set 5: Super Bowl

Opener

Complete this table. Don't worry, none of it is in base 2. Long division is fun! While it's a good idea to split the work among one another, please don't use any technology in this work or you may miss some big ideas.

We are the PCMI Shufflin' Crew. Shufflin' on down, doin' it for you.

Jokes so bad we know we're good. Blowin' your mind like we knew we would.

You know we're just shufflin' for fun, struttin' our stuff for everyone.

Fraction	Decimal representation	# of repeating digits
1/3	$0.\overline{333}$	1
1/5	0.2	n/a
1/7	$0.\overline{142857}$	6
1/9		
1/11		
1/13		
1/15		
1/17		
1/19		
1/21		
1/23		
1/25		
1/27		

Important Stuff

1. **a.** When doing long division, how can you tell when a decimal representation is about to terminate?
   b. How can you tell when a decimal representation is about to repeat?

In other words, when do you get to say "I'm done!" with these problems? I suppose you could really say that right now, but that's no fun, is it? Get back to work!!

2. Explain why the decimal representation of $\frac{1}{n}$ can't have more than n repeating digits.

Is there a better upper bound than n digits?

3. Find all solutions to each equation in mod 10.

In *mod 10*, the only numbers are 0 through 9. For example, 6 + 5 = 1. Fridge's number changes to 2, but McMahon gets to keep his 9.

a. $0x = 1$
b. $1x = 1$
c. $2x = 1$
d. $3x = 1$
e. $4x = 1$
f. $5x = 1$
g. $6x = 1$
h. $7x = 1$
i. $8x = 1$
j. $9x = 1$

4. **a.** Complete this multiplication table for mod 8 arithmetic.

In *mod 8*, Fridge would be especially unhappy to see his number changed to 0, since he'd be stuck wearing the same number as the punter with the cowbell and Panama hat.

×	0	1	2	3	4	5	6	7
0								
1								
2				6				
3	0							
4								
5								
6			4					
7								1

b. How many 1s did you see in your table above?

Don't include the ones on the sidelines.

5. Numbers can multiply with other numbers to make 1! It happens, but not always. Whenever this happens, both numbers are called *units*.

a. If you're working just with integers, what numbers are units?

When working with integers only, 5 is not a unit, since $5 \times \frac{1}{5} = 1$. But . . .

b. If you're working just with rational numbers, what numbers are units?
c. If you're working in mod 10, what numbers are units?
d. List all the units in mod 8.
e. List all the units in mod 15.

The "Shufflin' Crew Band" and "Shufflin' Crew Chorus" do not count as additional units.

6. Pick some more mods. Try to determine rules for what numbers are units in mod m, and how many units there are. Keep picking more mods until you have a feel for it.

"A Feel For Units" was narrowly rejected as the title for Chaka Khan's greatest hit.

Neat Stuff

7. What size decks will get restored to their original order after exactly 10 Thursday-style shuffles (and not in any fewer number of shuffles)?

I'm Samurai Mike I stop'em cold. Part of the defense, big and bold.

I've been jammin' for quite a while, doin' what's right and settin' the style.

Give me a chance, I'll rock you good, nobody messin' in my neighborhood.

(This man went on to coach the 49ers.)

8. **a.** How many units are there in mod 9? Call this number "Bond".

b. Build a multiplication table for mod 9 *but only include the units*. This multiplication table's size will be Bond-by-Bond.

c. Shuffle an 8-card deck using Thursday's shuffle style, and list the positions of all the cards at each shuffle. Look for something interesting!

Fun Fact: Da Bears were *nominated for a Grammy award* for their performance, and they remain the only professional sports team with a Top 41 hit single. (Wait, there's a Top 41 now?)

9. Can a power of 2 be a multiple of 13? Explain.

10. Multiply out these expressions.

$$(2^a - 1)(1 + 2^a + 2^{2a} + 2^{3a} + \cdots + 2^{(b-1)a}) = ?$$
$$(2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + \cdots + 2^{(a-1)b}) = ?$$

What does that tell you about $2^n - 1$ when n has factors?

11. Find the three prime factors of $2^{14} - 1$ astoundingly quickly, by hand.

12. That thing this morning. How'd we do that?

How'd we do what? You know what. The thing, with the thing.

13. With one per table, people picked 14 cards today for our magic trick.

a. Why isn't it a good idea to ask the question "What is the probability that at least two groups picked a duplicate card?"

b. If you draw 14 cards from a deck, with replacement, what is the probability that you pick a duplicate card?

What is the probability that the New England Patriots won the 1985 Super Bowl?

14. What size decks will get restored to their original order after exactly 14 Thursday-style shuffles (and not in any fewer number of shuffles)?

I'm mama's boy Otis, one of a kind. The ladies all love me for my body and my mind.

I'm slick on the floor as I can be, but ain't no sucker gonna get past me.

15. Determine all deck sizes that can be restored to their original order in 15 or fewer shuffles, and the specific number of shuffles needed for each.

16. Nicole, Amanda, Peggy, and Shaffiq are waiting in line, wondering if they can get to any of their 24 possible arrangements through these two rules:

- The person in the back of the group may jump to the front: $NAPS \Rightarrow SNAP$
- The person in the third position may switch with the person in the first position: $NAPS \Rightarrow PANS$

Can all 24 possible arrangements be made? Make a graph illustrating the options.

Oh, SNAP!

17. *It's p-adic number time!* Here are two interesting 10-adic numbers, and we're only going to show you their last six digits. (There are more digits to the left, feel free to try and figure out what they are.)

$$x = ...109376.$$

$$y = ...890625.$$

a. Calculate $x + y$.
b. Calculate the product xy.
c. Calculate x^2 and y^2.
d. How *crazy* are the p-adic numbers?!

Hey I'm Bowen, from EDC. I write books called CME.

I stay up late most every night, writin' problems that come out right.

I add numbers fast, just like magic, but my hairline is getting tragic.

You all got here on the double, so let's all do the PCMI shuffle . . .

Willie Nelson? Patsy Cline? Aerosmith? Britney Spears? Gnarls Barkley? Eddie? Horse? Madonna is crazy for $U(n)$.

Tough Stuff

18. Let n be an integer. Let $U(n)$ be the set of all deck sizes that are restored to its original order after exactly, and no fewer than, n Thursday-style shuffles. (You can disregard trivial decks with 0 or 1 cards.) In Problem 7, you calculated $U(10)$.

a. Prove that $U(n)$ is never empty: there is always some deck for which n shuffles is the lowest possible number.
b. For which n does $U(n)$ contain only one element?

19. **a.** For $n = 1$ through $n = 7$, find all n-digit numbers who last n digits match the original number. For example, $25^2 = 625$, ending in 25.
b. Find a connection between this and the p-adic numbers, or decide that there is no such connection.

My name's Darryl, I'm from LA. I work with math most ev'ry day.

I love to laugh, I love to eat, my Mathematica programs can't be beat.

I've taught kids of every age, now I'm in Park City writin' page by page.

I'm not here to fuss or fumble, I'm just here to do the PCMI Shuffle . . .