Problem Set 6: Cupid

Pre-Important Stuff

Wait what, *Pre-Important* Stuff? We're pregaming and it's only 8 am, that's big trouble.

Don't forget tonight is *Pizza and Problem Solving*. You'll have a good time. Or, at a minimum, pizza.

1. **a.** Find the repeating decimal for $\frac{1}{41}$. List all the remainders you encountered during the long division, starting with 1 and 10.

b. Write $\frac{100}{41}$ as a mixed number.

c. Find the repeating decimal for $\frac{18}{41}$. List all the remainders you encountered, including 1 and 10.

d. Find the repeating decimal for $\frac{1}{37}$. List them remainders!

e. Find the repeating decimal for $\frac{1}{27}$. Coolio.

No, it's not Coolio, the guy's name is Cupid. He even named the dance after himself, which seems a little presumptuous.

Opener

Complete this table. Splitting up the work is a great idea, but please do it without fancy spreadsheets or computer programs.

You're going to have to walk it by yourself, walk it by yourself.

n	Powers of 10 in mod n	Cycle Length
41		
37		
3		
7		
9		
11		
13	$1, 10, 9, 12, 3, 4, 1, \ldots$	6
17		
19		
21		
23		
27		
29		

→ Reminder: You don't have to calculate 10^5 to figure out the 4 in this row. Since you already know the previous 3 is equal to 10^4 in mod 13, you can use $3 \times 10 = 30 = 4$ mod 13. It's super helpful! Also, be careful of Cupid's arrows. This arrow points to the right, to the right.

Important Stuff

2. Consider the numbers n in the opener. Find all n among the list that are . . .

a. . . . factors of 9.

b. . . . factors of 99 but not of 9.

c. . . . factors of 999 but not of 9 or 99.

d. . . . factors of 9999 but not of 9, 99, or 999.

e. . . . factors of 99999 but not of 9, 99, 999, or 9999.

f. . . . factors of 999999 but not of . . . alright already.

What's up with that?

Ferris has been absent 9 times. 9 times? NINE TIMES.

I got 99 factored, and 7 ain't one.

Appropriate Beatles song: Number Nine.

Appropriate Nine Inch Nails song: 999999.

Ooooo weeeee, what up with that, what up with that!

3. Calculate each of the following. You may also want to look back at Problem Set 5.

a. $999999 \div 7$
b. $999999 \div 13$
c. $99999 \div 41$
d. $999999 \div 37$

$\Rightarrow$ Only five 9s this time! Cupid's arrow still points to the right, to the right.

4. Complete this table. A number x is a *unit* in mod n if there is a number y such that $xy = 1$. Yesterday we noticed that this is also the list of numbers in mod n that have no common factors with n.

You can say x is *relatively prime* to n, which is totally different than saying that x is *optimus prime*.

n	Units in mod n	# units in mod n
8	$1, 3, 5, 7$	4
15	$1, 2, 4, 7, 8, 11, 13, 14$	8
25		
30		
49	*too many units*	

$\leftarrow$ Was that just a *Sneakers* reference? I guess it is now! To the left, to the left.

5. How many units are there in mod 105? Counting them all would be a little painful.

6. Karen points out that the list of units in mod 15 contains 2 and 7, and $2 \times 7 = 14$ is also a unit.

a. Solve $2a = 1$ and $7b = 1$ in mod 15.
b. What is the value of $14ab$ in mod 15?
c. Explain why, if x and y are units in mod 15, then xy is also a unit.

7. Kieran points out that the list of units in mod 15 includes powers of 2: $1, 2, 4, 8$.

- **a.** Write a complete list of all the powers of 2 in mod 15. OK!
- **b.** Explain why, if x is a unit in mod 15, then x^2 is also a unit.
- **c.** Same for x^p for any positive integer power p.
- **d.** Why won't there just be billions of units if you can take any unit to *any* power p and make another one?

OK, Cupid? OKCupid's website says "We use math to get you dates"; its founder has a math degree. *The More You Know . . .*

Neat Stuff

8. Donna asks what size decks will get restored to their original order after exactly 10 Thursday-style shuffles (and not in any fewer number of shuffles).

9. Jen asks for what n does the decimal expansion of $\frac{1}{n}$ have an immediate repeating cycle of 10 digits and no fewer?

For example, $\frac{1}{15} = 0.0\overline{6}$ does *not* immediately repeat.

10. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be an ordered list in mod 9.

- **a.** Find a number M such that $2M = 1$ in mod 9.
- **b.** Calculate $M \cdot S$ in mod 9. *Keep everything in its original order.*
- **c.** Calculate M^2S in mod 9.
- **d.** Calculate M^kS in mod 9 for all k until something magical happens, then look back to see just how magical it is.

$\Leftarrow$ M^2S can also be used to send pictures to someone's phone. For large k, M^kS indicates that several people enjoyed a meal. To the left, to the left.

11.
- **a.** What are the units in mod 11?
- **b.** Which of the units in mod 11 are perfect squares? Which are not perfect squares?
- **c.** What are the units in mod 13?
- **d.** Which of the units in mod 13 are perfect squares? Which are not perfect squares?

In mod 11, 5 is a perfect square because $4 \cdot 4 = 5$. It's not the same in mod 13!

12.
- **a.** There are 16 units in mod 17. Of them, make a list of the eight that are perfect squares and the eight that are not.
- **b.** If x and y are perfect squares, is xy a perfect square... always? sometimes? never?

c. If x is a perfect square and y isn't, is xy a perfect square... always? sometimes? never?

d. If neither x nor y is a perfect square, is xy a perfect square... always? sometimes? never?

13. In shuffles, there are values of k for which there is only one deck size that restores in k perfect shuffles and no fewer. Are there values of k for which there is only one denominator n such that $\frac{1}{n}$ has a k-digit repeating decimal?

I get stupid. I shoot an arrow like Cupid. I'll use a word that don't mean nothin', like looptid. Hey, that guy named the dance after himself, too!

14. A *repunit* is a number made up of all ones: 11111 is a repunit. Investigate the prime factorization of repunits, and determine the values of k for which the k-digit repunit is prime.

15. Hey, we skipped $n = 15$ in the opener. What's up with that? Well . . .

a. What is the length of the repeating portion of the decimal representation of $\frac{1}{15}$?

b. What is the smallest positive integer n for which $10^n = 1$ in mod 15?

c. Aren't your answers for the previous two problems supposed to match? Figure out what's going on and rectify the situation.

I said, ooooo weeeee, what up with that, what up with that! Ooooo weeeee, what is *up* with that!

Well, looks like we're out of time.

Tough Stuff

16. **a.** Expand $(x-1)(x-2)(x-3)\cdots(x-6)$ in mod 7.

b. Calculate $1^6, 2^6, 3^6, \ldots, 6^6$ in mod 7.

c. For p prime and $x \neq 0$, prove $x^{p-1} = 1$ in mod p.

17. Turns out you can move the top card in a 52-card deck to any position, with a lot less than 52 shuffles. You just need a sequence of both Monday-style and Thursday-style shufflin' shufflin'. Find a method for moving the top card to any desired position in the deck with six or fewer shuffles.

Turns out lots of people get to name dances after themselves, including Dougie, Hammer, Urkel, Freddie, Macarena, Batman, Bartman, Pee-Wee, Ben Richards, and, of course, Carlton.