Problem Set 9: iPod

Opener

Use this set of equations to find the base-2 "decimal" for $\frac{5}{51}$.

$$\begin{aligned} 5 &= 0 \cdot 51 + 5 \\ &\vdots \\ 10 &= 0 \cdot 51 + 10 \\ 20 &= 0 \cdot 51 + 20 \\ 40 &= 0 \cdot 51 + 40 \\ 80 &= 1 \cdot 51 + 29 \\ 58 &= 1 \cdot 51 + \\ &= \quad \cdot 51 + \\ &= \quad \cdot 51 + \\ &= \quad \cdot 51 + 5 \end{aligned}$$

It's Friday! Gotta make my mind up, which seat can I take?

Thank goodness there are no giant tables on this problem set. Let's celebrate! As is customary, we gotta get down. See Carl perform as DJ Monochromatic Rectangle, 8 p.m. at Sidecar.

What would William Shatner yell if he was upset with the pedagogical style of a tutoring website? Answer later.

Important Stuff

1. **k.** John demands you write all the units in mod 51.

a. Write all the powers of 2 in mod 51, starting with 1.

l. Take all the powers of 2 in mod 51 and multiply them by 2. What happens?

i. Take all the powers of 2 in mod 51 and multiply them by 5. What happens?

n. Take all the powers of 2 in mod 51 and multiply them by 17. What happens?

Dave and Brian McKnight both agree that the list of powers ends back at 1.

Like Anthrax, you're caught in a mod! Once you're in a pattern you are stuck. It can't just go bipolar on you and change.

2. **a.** Without converting to a fraction, find a base-10 decimal such that

$$0.\overline{291594} + 0.\overline{mariah} = 1$$

It's like Mariah's "Get Your Number".

n. Without converting to a fraction, find a base-2 decimal such that

$$0.\overline{011001} + 0.\overline{rachel} = 1$$

a. In the obnoxious table, how do the base-2 expansions of $\frac{1}{51}$ and $\frac{50}{51}$ compare?

Wait, there's an obnoxious table? Oh no. *NOOOOOO!* Ask Bowen and Darryl for it.

3. Complete the obnoxious table. Be lazy and avoid the use of techmology.

Oh, bother.

4. Describe any patterns you notice in the obnoxious table. What fractions form "cycles" that use the same numbers in the same order? Write out each cycle in order, and find all the cycles.

5. Here is our happy little 52-card deck doing happy little out-shuffles, just like the very first day.

`http://tinyurl.com/pcmi52cards`

Follow some cards. Follow some remainders. Figure out how you can use the cards to find the *entire base-2 expansion* for a fraction in the form $\frac{n}{51}$.

Steven Tyler: "After some long hard thoughts, I've decided it's time to let go of my mistress 'American Idol'. It was over-the-top fun, and I loved every minute of it. Now it's time to bring Rock Back. ERMAHGERD." The guy actually wrote *ERMAHGERD* . . . in all caps . . . in a press release. There's only one word you can say to react to that.

Neat Stuff

6. Let's look at some eight-digit numbers in base 2.

- **a.** What number is 00011001_2 in base 10?
- **s.** One of the fractions in the obnoxious table has a base-2 expansion of $0.\overline{00011001}$. Figure out what fraction it is without looking at the obnoxious table, and without staring at cards like crazy.
- **h.** Start over with 00101101_2 and $0.\overline{00101101}$.
- **t.** Try it again with 11110000_2 and $0.\overline{11110000}$.
- **e.** What are the largest and smallest possible values of an eight-digit number in base 2?
- **n.** Can you explain why all the entries in the obnoxious table turn out to be multiples of [REDACTED]?

Cupid loves you! To the right . . .

Sorry, the multiplier's been redacted, just like last night's version of YMCA. "It's fun to say Y Y Y Y"? Seriously?

7.

- **z.** The base-2 number $110010ab_2$ is a multiple of 5. What are the missing digits?
- **a.** The base-2 number $001101ab_2$ is a multiple of 5. What are the missing digits, and what multiple of 5 is it?
- **c.** Make your own: pick the first six digits of a base-2 number and try to find the missing digits.
- **k.** That thing we did. How'd we do that?

You know, that thing. While cryptic, this still makes more sense than the lyrics to Mmmbop.

8. Based on the card animations, what do you get for the base-2 expansion for $\frac{51}{51}$? Interesting.

9. **j.** Take all the powers of 2 in mod 51 and multiply them by k. That was fun. Whee! It will be helpful, we promise.

Stay. You only hear what you want to.

o. A 52-card deck returns to its original position in 8 out-shuffles. But some cards return sooner. Find an equation that would be true for any card k that returns to its original position after 2 shuffles, then solve it.

Zero bottles of beer on the wall, zero bottles of beer. You take one down and pass it around, 50 more bottles of beer in mod 51.

s. Find an equation that would be true for any card k that returns to its original position after 3 shuffles, then solve it.

h. FOUR!

That's not even a question. But don't worry, be happy.

10. Use ideas from Problem 9 to show that if u is a unit, then u must be part of a cycle that has the same length as the cycle containing 1.

11. A 36-card deck returns to its original position in 12 out-shuffles. Determine all the cards that return sooner, and the cycle length of each.

12. Here are some shuffle animations. Triple shuffles!!! We promise they're cool.

`http://www.tinyurl.com/27cards`

Watch that wobble, see that wiggle, taste that jiggle. See, aren't you glad now we didn't choose the theme "Every Day We're Jello Pudd-ing?" Actually that sounds pretty awesome.

Go figure stuff out. Decimals in base 3, powers of 3 (in what mod?), cycles, magic tricks, all that jazz. What stays the same? What changes?

13. Determine all cycles of cards in the triple shuffle, and the cycle length of each. Given the number of units in the mod, what cycle lengths are possible?

It continues to be the case that you are shuffling in a typical siderial period.

14. **a.** Find a number x so that $x = 1$ in mod 11 and $x = 0$ in mod 13.

b. Find a number y so that $y = 0$ in mod 11 and $y = 1$ in mod 13.

c. Find a number z so that $z = 5$ in mod 11 and $z = 6$ in mod 13.

KHAAAAAANNNNNNNNN!

15. Find a number M so that $M = 2$ in mod 3, $M = 3$ in mod 5, $M = 4$ in mod 7, and $M = 5$ in mod 11. Try

this without techmology and with as little guesswork as possible!

Will computahs ever be able to work out what 999999999999999999 multiplied by 9999999 ... yo I haven't finished you don't know what I was going to say ... 99899999999.889988999 9999999999 is?

16. Look back at the table from Set 8 you made for the fractions from $\frac{1}{21}$ through $\frac{20}{21}$. All the digits from 0 through 9 are used, and there are 120 total digits in the repeating parts of the expansions.

a. Which digit is used most frequently in the decimal part of the expansions? Which digit is used least frequently?

b. Does this happen in other mods? Other bases?? So cool.

Okay, it's still not cool, but cool by math standards. Not cool like Kool Moe Dee cool, but hey.

17. Figure out a way to shuffle cards so that you could use the cards' positions after each shuffle to read off base-10 decimal expansions of fractions.

18. Nine-year-old David Cole conjectured that if the period of the base-10 expansion of $\frac{1}{n}$ is $n-1$, then n is prime. Using the concept of units, prove David's conjecture, and discuss whether it might apply to other bases.

Tough Stuff

19. Prove that there are infinitely many primes p for which the decimal representation of $\frac{1}{p}$ has cycle length $p-1$.

According to Carol, the group is "going to do Bowen at 3:15" today. That's scary. Anyway, it's a brief tour of the *CME Project* curriculum series and some ideas about specific topics like graphing and factoring.

20. The Fibonacci numbers start with $F_0 = 0, F_1 = 1, F_2 = 1$, etc:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Show that if m is a factor of n, then F_m is a factor of F_n. For example, $F_7 = 13$ is a factor of $F_{14} = 377$.

21. Show that if F is a Fibonacci number greater than 2, then neither $F^2 - 1$ nor $F^2 + 1$ is prime.

22. Take a grid of circles, 12-by-21, and color each circle with one of four colors. Either find a coloring that does not produce a monochromatic rectangle – a rectangle with all four corners the same color – or prove that such a coloring is impossible.

Once again, Carl plays tonight at 8 at Sidecar! Be there or be rectangular.