# Problem Set 10: Under the $\mathbb{Z}$

## Opener

You have four cards, arranged this way: 1234. You can perform any number of in-shuffles and out-shuffles on them, in any order. Can you get to all possible arrangements of the four cards? If so, show how. If not, explain why not.

Just look at the cards around you, right behind the math-camp door. Such wonderful mods surround you, what more is you lookin' for!

### Important Stuff

**1.** Laura notices in the opener that if you know the first two of the four cards, you can determine the order of the last two cards.

- **a.** If the first two cards are 13, what is the order of the last two cards?
- **b.** If the first two cards are 34, what is the order of the last two cards?
- **c.** Can the first two cards ever be 14?
- **d.** What is going on?

The weird $\mathbb{Z}$ stands for the integers. It comes from the German word "zahlen", meaning "weird-looking Z", and was first used by Evil Emperor Zurg.

Don't forget, this Thursday night is the "Enchantment Under The $\mathbb{Z}$" dance! Be there, or be rectangular.

**2.**
- **a.** Follow the 6 of spades and the 8 of diamonds in a regular deck of cards through the sequence of out-shuffles.

  `http://tinyurl.com/6spades8diamonds`

  What do you notice?
- **b.** Compare the base-2 "decimal" expansions of $\frac{5}{51}$ and $\frac{46}{51}$. What do you notice?
- **c.** Find other pairs of cards with the same behavior.

In $\mathbb{Z}$ all the numbers happy, they glad 'cause it's normal math. Numbers in the mod ain't happy, they stuck in a looping path.

**3.** You have six cards, arranged this way: 123321. Cards with the same number are identical. You can perform any number of in-shuffles and out-shuffles, in any order. Can you get the first three cards to take on all six possible arrangements of 123? If so, show how. If not, explain why not.

One of the other arrangements is 231132. The first half tells you what the last half has to be, so you could just read it as 231.

**4.** Build an addition table for $\mathbb{Z}_6$. Wait what? Oh, that's just mod 6. Also, please would you kindly build a multiplication table for $\mathbb{Z}_9$.

Look, it's Andrew Ryan's favorite phrase. (Obscure, but also under the sea!)

| $(\mathbb{Z}_6, +)$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | | | | | |
| 2 | 2 | | | | | |
| 3 | 3 | | | | | |
| 4 | 4 | | | | | |
| 5 | 5 | | | | | |

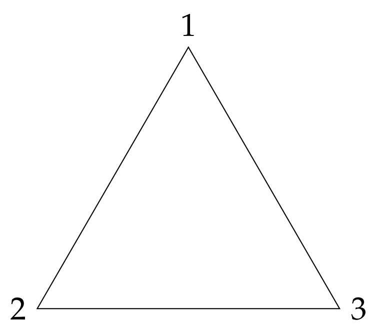
| $(\mathbb{Z}_9, \times)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | 0 | | | | | | | |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 2 | | 2 | | | | | | | |
| 3 | | 3 | | | | | | | |
| 4 | | 4 | | | | | | | |
| 5 | | 5 | | | | | | | |
| 6 | | 6 | | | | | | | |
| 7 | | 7 | | | | | | | |
| 8 | | 8 | | | | | | | |

**5.** Build a multiplication table for mod 9 that includes only its units. This table's size will be Bond-by-Bond. The set of units of $\mathbb{Z}_n$ is called $\mathbb{U}_n$.

A unit is a lot like a mushroom . . . no!!! A number $u$ is a *unit* if there is a number $v$ that solves $uv = 1$. The multiplication table should help, but we've also found another rule for deciding if a number in a mod was a unit.

If you're using the mathematical practices properly, you can use Problem 5 to model $\mathbb{U}_n$! A joke only teachers could love.

**6.** Here's an equilateral triangle:




Suppose you can perform any number and sequence of reflections or rotations, as long as you leave one of the corners of the triangle pointing up as shown.

**a.** Draw all six possible configurations of the triangle.

**b.** Build an operation table for transforming the equilateral triangle, where the operation is "then". For example, you could rotate the triangle 120 degrees counterclockwise, then reflect the triangle across its vertical line of symmetry. This combination of moves is equivalent to what single move? Do this to complete the table.

This table's size will also be Bond-by-Bond. One of the six options is "do nothing", leaving the triangle in its present orientation.

**7.** **a.** What is the *identity* for addition?

**b.** What is the identity for multiplication?

**c.** Why isn't 0 the identity for multiplication?

**d.** What is the identity for triangle transformation?

**8.** The *cycle length* of an element is the number of times you have to repeat its operation to get back to the identity. For example, in $\mathbb{Z}_6$ under addition, the cycle length of 4 is 3:

$$0 + 4 = 4 \text{ then } 4 + 4 = 2 \text{ and then } 2 + 4 = 0$$

**a.** Find the cycle length for all elements of $\mathbb{Z}_6$ under addition. One of the cycle lengths is 1!

**b.** Find the cycle length for all elements of $\mathbb{U}_9$ under multiplication. Notice anything interesting?

**c.** Find the cycle length for all elements of the triangle transformations. Notice anything interesting?

The cycle length of adding 2 in $\mathbb{Z}_{10}$ is 5. The cycle length of multiplying by 2 in $\mathbb{Z}_{51}$ is 8. The cycle length of Dory is about 10 seconds.

And then? *No and then!* And then??

So, it's 1 factorial, or just 1?

I'm crazy for $\mathbb{U}_n$!

I noticed that Triangle Man hates Person Man. They might have a fight.

### Neat Stuff

**9.** Go back to the equilateral triangle. Suppose you're only allowed 120-degree counterclockwise rotations and reflections across the vertical line of symmetry. Can you get to all six possible configurations using only these moves? If so, show how. If not, explain why not.

What is a French chef's favorite statistical distribution? *Le Poisson!* Hee hee hee, haw haw haw!

**10.** Are in-shuffling and out-shuffling commutative? Explain.

**11.** Are the six triangle transformations commutative? Explain.

**12.** Try Problem 6 again with a tetrahedron. You'll have to figure out how many possible configurations there are, then build the operation table.

**13.** If $x$ is an element of $\mathbb{Z}_n$, show that there must *either* be an element $y$ that solves $xy = 1$, *or* an nonzero element $z$ that solves $xz = 0$, but not both.

We got the spirit, you got to hear it, under the $\mathbb{Z}$.

The two is a shoe, the three is a tree.

The four is a door, the five is a hive.

The six is a chick, the seven's some surfin' guy . . . *yeah*

The eight is a skate, the nine is a sign.

And oh that zero blow!

**14.** **a.** Find all solutions to $x^2 = 1$ in $\mathbb{Z}_{105}$. There are a lot of them, and busting 105 into little pieces may help.
**b.** Use completing the square (!) to find all solutions to $x^2 + 24 = 10x$ in $\mathbb{Z}_{105}$.

Near a beach, there was a dangerous tree with a beehive in it. A surfer rode a big wave all the way onto land and crashed into the tree. The bees got mad and chased him! He had to dive back into the water to escape from the bees.

Tree × hive × surfin' equals wanna dive!

**15.** We now know that 52 shuffles will restore a deck of 52 cards using in-shuffles. This is true because $2^{52} = 1$ in mod 53. Suppose you wanted to prove this to a friend, but you only have a simple calculator that can't calculate $2^{52}$ exactly. Find a way to calculate $2^{52}$ in mod 53 using as few operations as possible and just a simple calculator. What other powers of 2 would you need to calculate to ensure that it takes 52 shuffles to restore the deck and not some smaller number?

**16.** **a.** Without a calculator, determine the number of in-shuffles it will take to restore a "double deck" (104 cards) to its original state.
**b.** How many out-shuffles will it take?

What mod is it for each of these questions?

## Tough Stuff

**17.** Without a calculator, determine the number of in-shuffles it will take to restore a deck of 20,000 cards to its original state.

Oh man. 20,000 cards under the $\mathbb{Z}$? Somebody call Nemo.

**18.** Jay says there's this box. It's got integer dimensions, like 6-by-8-by-10 but not.

**a.** All three of the diagonals on the faces of the box also have integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.
**b.** Additionally, the space diagonal (from one corner of the box to the other corner in eye-popping 3D) also has integer length. Find a possible set of dimensions for the box, or prove that no such box can exist.

*Under the Sea 3D* is now playing at a theater near you! Well, if by *near* you mean the Omnitheater at the Science Museum of Minnesota, which is literally the nearest theater playing that. While you're there, visit the new *Math Moves!* exhibit with this totally sweet perspective-drawing thing.