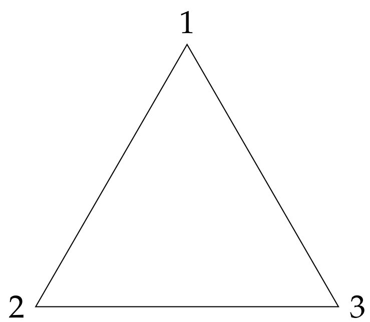
Problem Set 11: 𝕌 Can't Touch This

Ma, ma, ma, ma math it hits me so hard
Makes me say "Er, mah gerd"
Thank 𝕌 for asking me
To take triangles and shuffle their feet

Opener

Let's revisit the transformations of the equilateral triangle from the last problem set. But this time, we'll use wacky *permutation notation* to describe the six transformations you can perform. Here are the six transformations.



Permutation	Transformation on the triangle
()	Do nothing
(1 2 3)	
(1 3 2)	
(1 2)	
(1 3)	
(2 3)	

(1 2 3) means 1 goes to 2, 2 goes to 3, 3 goes to 1.

(1 3) means 1 goes to 3, 3 goes to 1.

Now complete this operation table, where the operation is "then". For any cell in the table, perform the transformation that labels the row first, then the transformation that labels the column. Write the transformation that is equivalent to the combination of those two transformations.

"then"	()	(1 2 3)	(1 3 2)	(1 2)	(1 3)	(2 3)
()						
(1 2 3)						
(1 3 2)						
(1 2)						
(1 3)						
(2 3)						

Give me a square or rectangle
Findin' symmetries at every angle
2 . . . Legit! Find them all or
𝕌 might as well quit

That's word because 𝕌 know . . .

Important Stuff

1. Marina hands you this 8-card deck:

0	1	2	3	4	5	6	7

a. Using this deck, explain why an out-shuffle can be represented by this permutation notation:

$$(1\,2\,4)(3\,6\,5)$$

Stop! Shuffle time. Hey wait, why isn't there a 0 or a 7 in this notation?

b. Write permutation notation for an in-shuffle using this deck.

c. Use the permutation notation to determine the number of out-shuffles needed to restore an 8-card deck to its original position, and the number of in-shuffles needed.

2. Build an addition table for $\mathbb{Z}_4$. Oh, that's just mod 4. Also, build a multiplication table for $\mathbb{U}_5$. Remember $\mathbb{U}_5$ contains all the units in mod 5, which are 1, 2, 4 and 3. What's with the weird order? That's word.

What's a $\mathbb{Z}_4$? For ending the alphabet, silly. $\mathbb{U}_5$ at that table had better get back to work.

$(\mathbb{Z}_4, +)$	0	1	2	3
0				
1				
2				
3				

$(\mathbb{U}_5, \times)$	1	2	4	3
1				
2				
4				
3				

3. Leah says that $\mathbb{Z}_4$ is "just like" $\mathbb{U}_5$. Say *what*??

I'm just like $\mathbb{U}$! I'm just like $\mathbb{U}$ . . .

4. In the previous problem set, you built the addition table for $\mathbb{Z}_6$ and the multiplication table for $\mathbb{U}_9$. Can you make those two tables match by pairing the numbers up in some way? Explain.

5. Now compare the addition table for $\mathbb{Z}_6$ with the operation table for the triangle transformations. Can you make those two tables match by pairing the entries in some way? Explain!!!

Use cycle lengths to help you! The *cycle length* is the number of times an element's operation repeats until the identity appears. For addition, solve $x + x + \cdots + x = 0$ and figuring out how many x's are needed. For multiplication . . . and for triangle transformation . . .

6. **a.** Find the cycle length for each element of $\mathbb{Z}_4$ under addition.

b. Build a multiplication table for $\mathbb{U}_8$, then find the cycle length for each element under multiplication.

7. Can you match the addition table for $\mathbb{Z}_4$ and the multiplication table for $\mathbb{U}_8$ by pairing numbers up in some way?

8. Look at the table on today's handout. Find an important relationship between the number of units in mod n and the cycle length of 2^k in mod n that is consistently true. Do it again between the number of units in mod n and the cycle length of 10^k in mod n.

Are $\mathbb{U}$ glad $\mathbb{U}$ didn't have to fill this out?

Here are some unimportant relationships between the columns:

"The columns all contain numbers."

"The columns all contain positive numbers."

"The columns all contain one-, two-, and three-digit numbers."

"None of the entries in any column is the number 8675309."

Neat Stuff

9. Write the 52-card out-shuffle using permutation notation, and use it to explain why the deck is restored in eight out-shuffles.

10. **a.** In $\mathbb{U}_9$ there are six numbers: 1, 2, 4, 5, 7, 8. The number 2 is called a *generator* of $\mathbb{U}_9$ because every number is a power of 2: $1, 2, 4, 8, 7, 5, 1, \ldots$. Which other numbers in $\mathbb{U}_9$ are generators?

b. What are the generators in $\mathbb{Z}_6$ under addition? Instead of using powers (repeated multiplication) of numbers, you will need to use repeated addition.

c. How can generators help you match the tables for $(\mathbb{U}_9, \times)$ and $(\mathbb{Z}_6, +)$?

It tours around $\mathbb{U}_9$, from 1
then all the way
It's 2, go 2, generate it now
2
That's all there is to say

11. What are the generators in $\mathbb{Z}_7$ under addition? In $\mathbb{Z}_8$? Neat.

Of course it's neat, this is Neat Stuff.

12. Describe some conditions under which you can say that the tables for two operations can *definitely not* be matched. Find more than one condition!

I told you, homeboy, $\mathbb{U}$ can't match this!

13. Find all $n \neq 8$ so that the table for $\mathbb{U}_n$ under multiplication can match the table for $\mathbb{U}_8$ under multiplication.

14. Make a table of the number of generators of $\mathbb{U}_n$ for different n. What patterns do you notice?

I love $\mathbb{U}$, $\mathbb{U}$ love me. Oh no.

15. **a.** Find the cycle lengths of 3 and 5 under multiplication in $\mathbb{U}_{13}$ and explain why each is *not* a generator.

b. What are the generators of $\mathbb{U}_{13}$?

Never mind, I'll find someone like $\mathbb{U}$ . . .

c. So 3 and 5 are not generators. However, Ruth says that any number in $\mathbb{U}_{13}$ can be written as $3^a \cdot 5^b$ for some a and b. Show that she's right!

How many elements are in $\mathbb{U}_{13}$? How long are the cycles of 3 and 5? Hmm.

16. Show that no single transformation is a generator for the six triangle transformations, but that it is possible to choose two transformations that generate all six transformations together.

Yo, sound the bell, school is in, sucka!

17. Here are six functions.

- $m(x) = 1 - \dfrac{1}{1-x}$
- $o(x) = 1 - x$
- $n(x) = 1 - \dfrac{1}{x}$
- $i(x) = x$
- $c(x) = \dfrac{1}{x}$
- $a(x) = \dfrac{1}{1-x}$

a. Build an operation table for working with these six functions, where the operation is "composition". For example, $n(o(x)) = m(x)$.

b. Which, if any, other tables can match this table by pairing the entries in some way?

Yummy: $n \circ o = m$! It's too bad Hammer never had his own cereal. He did get a cartoon show, though.

18. **a.** Find the number of out-shuffles that it will take to restore a deck of 90 cards. Do this without a calculator and using the most efficient method possible, given what you learned in Problem 8.

b. Find the number of repeating digits in the decimal expansion of 1/73 without long division or a calculator.

Stop! Shuffle time.

$\mathbb{U}$ might be more efficient if $\mathbb{U}$ look back at Problem 15 from Problem Set 10.

I guess the change in my pocket wasn't enough, I'm like, forget $\mathbb{U}$.

Tough Stuff

19. If p is prime, prove that every $\mathbb{U}_p$ has at least one generator.

Don't worry, tomorrow's problem set is *not* titled "We $\mathbb{R}$ Who We $\mathbb{R}$".

20. If p is prime, prove that every $\mathbb{U}_p$ has exactly ____________ generators. Hm, looks like we left that spot blank.

21. Find all *composite* n for which $\mathbb{U}_n$ has generators.

Who are $\mathbb{U}$? Who who, who who?

22. If p is prime, find some general conditions under which the number 2 is *definitely* or *definitely not* a generator in $\mathbb{U}_p$.

If you missed any of today's comments, look them up on $\mathbb{U}_{\text{tube}}$.