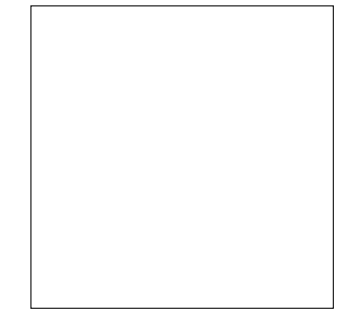
Problem Set 12: Like a d6

Opener

It's squaresville today. What are the *eight* transformations that you can perform on the square below so that it still fits in this space?



Permutation	Transformation on the square
$()$	Do nothing
$(1\ 2)(3\ 4)$	Reflection across vertical axis

This episode of Square One Television is brought to you by the identity letter e, and by the number e.

Cut out your own square and label its corners on both sides as shown in the diagram. A special prize will be given to the most beautiful square!

Poppin' yogurts in Grub Steak, makes us snicker

When we think about some math, it gets twittered

Sippin' water while we learn . . . some math tricks

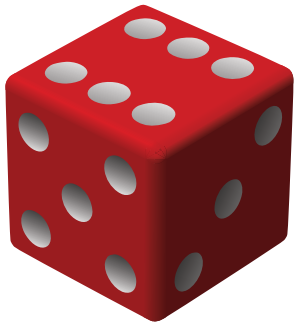
Now I'm countin' symmetries of a d6 . . .

Now complete this operation table, where the operation is "then". For any cell in the table, perform the transformation that labels the row first, then the transformation that labels the column. Write the transformation that is equivalent to the combination of those two transformations.

"then"	$()$							
$()$								

Important Stuff

1. Here is a picture of a d6, with the 6 facing up.



Like a d6, like a d6, now now now now now I'm feeling like calling it a number cube. Wait, that's probably not how it goes.

a. Describe all the ways you could move the d6 so that it keeps the 6 on top and still occupies the same space.
b. How many transformations keep the 6 on top?
c. How many orientations does the d6 have in total? You no longer have to keep the 6 on top.

We keep it real here at PCMI. All transformations should actually be performable, which rules out something you could do with a square or triangle. Like the square and triangle, the d6 has to end up in the same position, but the orientation could be totally different.

2. Build an operation table for your transformations from Problem 1a, where the operation is "then". You can use any notation that you like. How big should this table be? Where have you seen a similar operation table?

3. Look back at Problem 3 on Set 10. Six cards were arranged this way: 123321. Cards with the same number were identical. Look for the graph on today's handout. Each arrow in the diagram shows how you can get from one arrangement to another using a shuffle. Write an "I" next to each arrow corresponding to an in-shuffle and an "O" next to each arrow corresponding to an out-shuffle.

The "out shuffle" is the Monday shuffle with the stuck cards, and the "in shuffle" is the Thursday shuffle.

4. Laurie, Marian, and Nadine are waiting in line, wondering if they can get to any arrangement through these two rules:

- The person in the back of the group may jump to the front: LMN $\Rightarrow$ NLM
- The two people at the back of the group may swap places: LMN $\Rightarrow$ LNM

Can all six possible arrangements be made? Make or find a graph illustrating the options.

The set of all arrangements of LMN is called the Samsung Galaxy S3, or perhaps just S_3.

Swaps, swaps, swaps! Swaps swaps swaps!

5. Look back to your operation table from the Opener from Set 11.

- **a.** What is the "identity" in this operation table?
- **b.** Which transformations have inverses?
- **c.** For each transformation, how many times do you have to perform it to restore the equilateral triangle to its original state?
- **d.** Complete this sentence: "The cycle length of each transformation _____ the total number of transformations."

An element with an inverse has been called a *unit*.

Bazinga?

6. Titin is waiting in line while holding an equilateral triangle, wondering if she can get to any of the six orientations through these two transformations:

- A rotation: $(1\ 2\ 3)$
- A reflection: $(2\ 3)$

Can all six possible orientations be achieved? Make or find a graph illustrating the options.

. . . as you do. It's probably happened a few times here, actually.

7. Two groups are called *isomorphic* if there is a correspondence between them that matches their operations completely. Describe at least three isomorphisms you have found in this course so far, and at least two *non*-isomorphisms.

Drink it up, yeah, drink it up

When $(U_5, \times)$ around me it be acting like $(Z_4, +)$

Neat Stuff

8. The group S_4 is the set of all possible permutations acting on the numbers 1 through 4, where the operation between the permutations is "then".

- **a.** How many elements are in S_4? Explain how you know.
- **b.** Decide whether or not S_4 is isomorphic to the group of transformations of the square from the opener. What what!

9. A d4 is a tetrahedral die. Picture its faces numbered 1 through 4. As a shorthand, 4 will be represented by the word "fruit".

Worst shorthand ever?

a. Describe all the ways you could move the d4 so that it keeps the fruit on the bottom and still occupies the same space.
b. How many transformations keep the fruit on the bottom?
c. How many orientations does the d4 have in total? You no longer have to keep the fruit on the bottom.

Occupy d4!

Poppin' yogurts in Grub Steak . . .

10. A d8 is an octahedron. How many orientations does it have?

You shoulda had a d8!

11. How many orientations are there if your die is

a. . . . like a d12?
b. . . . like a d20?
c. . . . like a d10?

In the basement rollin' dice, I'm a wizard

When we play we think we fight giant lizards

Now now now now now now don't want my elf to die, roll a d20.

12. An element of a group is a *generator* if repeated operation of that element takes you through every element of the group.

a. Find all the generators for $(\mathbb{Z}_{12}, +)$ or explain why there aren't any.
b. Find all the generators for $(\mathbb{U}_{12}, \times)$ or explain why there aren't any.
c. Find a generator for S_3 or explain why there isn't one.

13. Sometimes $(\mathbb{U}_n, \times)$ has a generator, and sometimes it don't.

a. Under what conditions will $\mathbb{U}_n$ have a generator?
b. In terms of n, *how many generators* are there?

Sometimes you feel like a generator, sometimes you don't.

14. A d6 has the numbers 1 through 6 on it. How many *different* ways are there to put the numbers on a d6? By *different* we mean that there is no transformation taking one arrangement to another.

Roll your own d6, just not in Vegas.

Tough Stuff

15. **a.** Is it possible to generate S_4 using only in- and out-shuffles on cards numbered 12344321?
b. Is it possible to generate S_5 using only in- and out-shuffles on cards numbered 1234554321?