Problem Set 13: The Vertex Of Glory

Opener

On Monday, you made a list of all possible arrangements of the cards 1234 with an unlimited supply of in- and out-shuffles. One side of today's handout contains these eight arrangements. Draw arrows connecting the arrangements. Label each arrow with "I" or "O" to indicate whether the an in- or out-shuffle connects the two arrangements.

An *out-shuffle* keeps the top and bottom cards stationary, while an *in-shuffle* moves everything. I'm your biggest fan, I'm shufflin' until you love me.

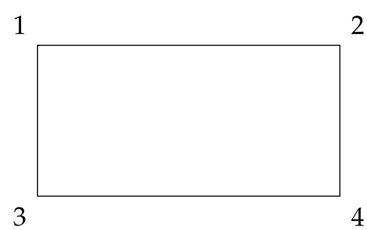
Important Stuff

1. Draw the eight orientations of the square from Set 12's opener. Use arrows in one color to connect two orientations if the $(2\ 3)$ transformation takes one orientation to the other. Use arrows in a different color to connect two orientations if the $(1\ 2\ 4\ 3)$ transformation takes one orientation to another. Don't draw arrows for other transformations. Notice anything?

No matter black, white, or beige. Don't be a drag. You're on the right track, baby.

2. Use your two diagrams to argue that the group generated by in- and out-shuffles on 4 cards is isomorphic to D_4, the group of symmetries of the square.

3. Here's a rectangle:



a. What are the four transformations you can perform on the rectangle so that it still fits in its space?

b. Complete an operation table for the rectangle, where the operation is "then".

c. Find another group you've worked with this week that also has four elements, but is definitely *not* isomorphic to it. Explain how you know they're not isomorphic.

d. Find another group you've worked with this week that has four elements and *is* isomorphic to the rectangle group.

Balderdash time! Six of the seven definitions below are false. Which is the right one?

Alex says that the D in D_4 stands for *Diophanteen*, the latest advance in hair-care technology.

Christina says that the D in D_4 stands for *denom-nomnator*, which is what happens when you can completely cancel out bottom of a fraction.

Dominic says that the D in D_4 stands for *Dominic*, duh.

Eric says that the D in D_4 stands for *de Bruin cycle*, which is how often UCLA wins football games.

Kaelin says that the D in D_4 stands for *de Mauve's Theorem*, which describes when mathematics can and can't be purple.

Gregory says that the D in D_4 stands for *Deerichlay*, which is a fancy wine from the French region of Bourbaki.

Rachell says that the D in D_4 stands for *dihedral*, meaning "two-sided".

4. **a.** A *d4* is a tetrahedral die. How many faces does a d4 have? Alright, not our most brilliant problem ever.
b. How many different orientations (aka symmetries) does a d4 have?

The d in d4 stands for *die*, meaning "die".

5. How many edges does a d4 have? Use this to verify the number of orientations you found in Problem 4.

6. Determine the number of different orientations for a d6. Find the number of different orientations separately using faces, then using edges, then using vertices.

I want your $\mathbb{Z}_{12}$ and I want your mod 10, you and me could write a bad conjecture.

7. Determine the number of different orientations for a d8. Find those different ways! Sweet.

8. . . . for a d12.

9. . . . for a d20.

Review Your Stuff

10. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 15 minutes** on this. Make sure your question is something that ∗**everyone**∗ at your table can do, and that you expect ∗**everyone**∗ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Is there really such a thing as a "self-reflective process of discovery"? Yes, there really is! Don't believe us? Ask Google, and put quotes around it.

There ain't no reason A and B should be alone

Today, yeah baby, today, yeah baby

I'm on the vertex of glory

And I'm hanging on a corner with you

I'm on the vertex, the vertex, the vertex, the vertex, the vertex, the vertex, the vertex

I'm on the vertex of glory

And I'm hanging on a corner with you! I'm on the corner with you!

Stupid Stuff

11. Draw a table.

12. Solve for x:

$$(P^3x + P^2x + M^4ah)^4 = \text{ can't read my } + x$$

Neat Stuff

13. The rectangle transformation group is usually called D_2, the triangle transformation group is D_3, and the square transformation group is D_4. Find some common Bond between these groups and their makeup.

There is no common Bond between Lady Gaga's groups and her makeup.

14. **a.** Show that $(1\ 2)(1\ 3) = (1\ 2\ 3)$.
b. Show that $(1\ 2)(1\ 3)(1\ 4) = (1\ 2\ 3\ 4)$.
c. What's the next one?
d. Pick apart $(1\ 2\ 4\ 8\ 7\ 5)$ into five transpositions.

Two-element thingies like $(1\ 2)$ are called *transpositions*.

15. Rewrite $(1\ 3)(2\ 3)(2\ 5)(1\ 2)(2\ 5)(4\ 5)$ using a single set of parentheses.

The answer is *not* (132325122545).

16. A permutation is called *even* if it can be broken into an even number of two-element transpositions, and called *odd* if it can be broken into an odd number of two-element transpositions. Problem 15 contains an even permutation.

a. Of the eight permutations for the square transformation group, how many are even? How many are odd?
b. What happens if you combine two even permutations?
c. . . . two odd permutations? One odd, one even?

I've had a little bit too much, mods
All of these problems start to rush, start to rush by
How does he shuffle the cards? Can't stop and think, oh man
Where's my table? I broke my lamp, lamp
Just math. Gonna be okay.

17. These are the 24 permutations making up S_4, all the ways to go from one ordering of 1234 directly to another:

• $()$	• $(1\ 2)$	• $(1\ 2\ 3)$	• $(1\ 2\ 3\ 4)$
• $(3\ 4)$	• $(1\ 2)(3\ 4)$	• $(1\ 2\ 4\ 3)$	• $(1\ 2\ 4)$
• $(2\ 3)$	• $(1\ 3\ 2)$	• $(1\ 3)$	• $(1\ 3\ 4)$
• $(2\ 4\ 3)$	• $(1\ 4\ 3\ 2)$	• $(1\ 4\ 3)$	• $(1\ 4)$
• $(2\ 3\ 4)$	• $(1\ 3\ 4\ 2)$	• $(1\ 3)(2\ 4)$	• $(1\ 3\ 2\ 4)$
• $(2\ 4)$	• $(1\ 4\ 2)$	• $(1\ 4\ 2\ 3)$	• $(1\ 4)(2\ 3)$

This is not the same as in- and out-shuffling cards 1234. This is like having cards 1234 and just directly sorting them any way you want it, that's the way you need it.

a. Find all the even permutations. How many are there?
b. For each even permutation, write the result when 1234 is transformed by the permutation.

Your work in Problem 14 will help you decide.

18. You may or may not have made a list of all possible arrangements of the cards 12344321 with an unlimited supply of in- and out-shuffles.

One side of today's handout contains these arrangements. Draw arrows connecting the arrangements. Label each arrow with "I" or "O" to indicate whether the an in- or out-shuffle connects the two arrangements.

In this arrangement, the paired cards are considered equivalent, and the first half of the deck tells you what the second half must look like.

19. **a.** What two permutations are represented by the shuffles in Problem 18? Are they odd or even?
b. Using a d4, find a way to reproduce these two permutations visually. One permutation keeps "1" in place, and another keeps "3" in place.

Fun fact: Darryl likes Lady Gaga because she has monsters!

20. Explain why the group generated by in- and out-shuffles on 12344321 is isomorphic to A_4, the group of symmetries of the tetrahedron.

Alyssa says the A in A_4 stands for *Alyssa*, duh! Ok, actually the A in A_4 stands for *alternating*, meaning that every other element in S_4 is included. Since S_4 has 24 elements, A_4 has 12. A_n is made up of all even permutations of S_n.

21. Find four functions whose group under composition is isomorphic to the rectangle transformation group.

22. It sure is interesting that the number of orientations of a d6 is the same as the number of orientations of a d8. That suggests these two transformation groups might be isomorphic. What do you think?

It's got to be yes, because otherwise the question wouldn't be here, right? Boom! QED. More mathematicians should drop their chalk like in *Drumline* when they finish a proof.

Tough Stuff

23. It is always possible to write a permutation without re-using a element. When the $n!$ permutations of S_n are written this way, what fraction use all n elements? For S_4, 9 of the 24 transformations use all four elements.

Stop askin', stop askin', I don' wanna think anymore!

24. George Sicherman discovered a *different* way to populate 2d6 with positive integers, so that the sums of the two d6 matched the usual distribution. Neither d6 has the usual 1-6 on it. Only positive integers are allowed, and repetition is allowed.

a. Figure out what numbers are on the Sicherman dice.
b. Find all possible "Sicherman-like" dice for the d4, d8, d12, and d20. There may be more than one possible answer, or none at all! Woo hoo ha ha ha.

2d6 means two six-sided dice.

Wait, that's not a Gaga lyric!