Problem Set 1: Fake It 'Til You Make It

Welcome to PCMI. We know you'll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. Here's a few things you should know about how the class is organized.

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.

Some problems are actually *unsolved*. Participants in this course have settled at least one unsolved problem.

- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Respect everyone's views.** Remember that you have something to learn from everyone else. Remember that everyone works at a different pace.
- **Teach only if you have to.** You may feel tempted to teach others in your group. Fight it! We don't mean you should ignore people, but don't step on someone else's aha moment. If you think it's a good time to teach your colleagues about Bayes' Theorem, don't: problems should lead to appropriate mathematics rather than requiring it. The same goes for technology: problems should lead to using appropriate tools rather than requiring them.

Strategically, natch.

- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out Important Stuff first. All the mathematics that is central to the course can be found and developed in the Important Stuff. *That's* why it's Important Stuff. Everything else is just neat or tough. If you didn't get through the Important Stuff, we probably noticed . . . and that question will be seen again soon. Each problem set is based on what happened in the previous set, and what happened in the previous *class*.

When you get to Problem Set 3, go back and read the introduction again.

What is the probability that you will remember to do this?

Opener

1. On a piece of paper, write the results of flipping a coin 120 consecutive times, 10 flips per row, writing heads as 1 and tails as 0.

   But wait: **don't flip coins**. *Fake* the data. Your goal is to write a highly believable fake with no help from anyone or anything. Don't use anything to generate the flips other than your brain. Make it look organized. Write your full name on the page.

One possible row is 1111000010: four heads, then four tails, then one head, then one tail.

2. On a different piece of paper, write the results of flipping a coin 120 consecutive times, 10 flips per row, writing heads as 1 and tails as 0.

   But wait: **flip coins**. Write down the actual results of flipping a coin 120 consecutive times. Don't use anything to generate the flips other than your coin. Make it look equally organized. Write your full name on this page, too, in the same way.

The goal is to make the two lists indistinguishable, except for the different flips.

3. Suppose you were handed two lists of 120 coin flips, one real and one fake. Devise a test you could use to decide which was which. Be as precise as possible.

Attend to PCMI!

4. **a.** Mark one of your real or fake pages with the letter A. Mark the other with letter B. *Be sure you know which is which.*
   b. Exchange lists with someone, ideally from another table. Use the test you developed in Problem 3 to decide which list is real and which is fake. Then, find out if you were right!
   c. Make better tests! Keep exchanging! More, more!

5. Talk amongst yourselves about what happened. What went wrong with the fakes? What tests were accurate, and what tests weren't? What would you do next time?

Important Stuff

There is more than one way to do this! Keep track of the assumptions you make. Consider comparing the different options here, and differences in corresponding results.

6. **a.** Find the probability that when you pick two integers between 1 and 5 (inclusive), they do not share a common factor greater than 1.

b. Repeat for picking between 1 and 6, 1 and 7, 1 and 8, 1 and 9.

7. After mysteriously flipping coins on a bus for an hour, Tina says that if a set of 120 flips has exactly 60 heads and 60 tails, it's *obviously* fake.

Where is Tina, anyway?

a. **In 10 seconds or less** estimate the proportion of fake data sets that have exactly 60 heads and 60 tails.

b. **In 10 seconds or less** estimate the proportion of real data sets that have exactly 60 heads and 60 tails.

8. Record your real and fake flipping data on this website:

`http://bit.ly/fakeflips`

Please be careful and accurate!

Neat Stuff

9. What's the probability that an integer picked from 1 to n is a perfect square if

a. $n = 20$?

b. $n = 200$?

c. $n = 2000$?

d. $n = 20000$?

e. What is happening "in the long run" (as n embiggens forever)?

10. Faynna offers you these two games:

Game 1: You roll a die four times. If you roll a six any of the four times, you win.

Game 2: You roll a pair of dice 24 times. If you roll boxcars (double sixes) any of the 24 times, you win.

Aside from the fact that Game 2 takes longer to play, which of these games would you rather play to win? Or do both games have the same chance of winning?

Problem 10 lies at the foundation of probability theory, and was originally solved by Pascal.

11. Brian, Brian, Marla, Jennifer, Moe, and Elmer go to dinner every night and play "credit card roulette": the waiter picks one of their six credit cards at random to pay for the meal.

a. What is the minimum number of meals it will take before each of them has paid at least once, and what is the probability of this occurring?

b. What is the maximum number of meals it will take before each of them has paid at least once? Uh oh.

c. What is the *mean* number of meals it will take before each of them has paid at least once?

Problem 11 was neither posed by nor solved by Pascal, who always paid in cash.

Tough Stuff

12. In Yahtzee, you get three rolls and you're looking to get all 5 dice to be the same number. You can "save" dice from one roll to the next. There are other goals, but it's not called Yahtzee for nothing.

a. Find the probability that if you try for it, you will get a Yahtzee of all 6s by the end of your third roll.

b. (*harder*) Find the probability that if you try for it, you will get some Yahtzee by the end of your third roll. Assume that you always play toward the nearest available Yahtzee: if your first roll is 2-3-3-5-6, keep the 3s and roll the rest.

Why *is* it called Yahtzee? Apparently the creators played it on their yacht, perhaps near the Tappan Zee Bridge.

13. What is the mean number of meals it will take before the six friends from Problem 11 each pay at least *twice*? At least n times?