Problem Set 5: Shell Game, Macklemore Edition

Opener

In today's game, the player had to decide whether each of the four popped tags was more or less than 99 cents. The prices were so close to 99 cents that the choices were 50/50.

Why such cheap prizes? Bowen and Darryl only had $20 in their pocket.

1. **a.** What is the probability that someone playing this game will get all four questions right?
b. Write out all 16 ways the game can go, using R for right and W for wrong.
c. What is the probability that a player correctly answers *exactly three* questions?
d. . . . two questions? one question? *No* questions?

2. Expand $(R+W)^4 = (R+W)(R+W)(R+W)(R+W)$ any way you like; you might practice on $(R+W)^2$ first. Notice anything?

Use the Nspire's `expand` command, or GeoGebra, or maybe Wolfram Alpha, or a pencil, or looking at Problem 1! Nspire instructions are at the end of today's problem set.

3. A player's chance of winning the big prize depends on how many tags they successfully pop.
a. If Caitlin pops one tag, what is her probability of winning the big prize?
b. Katie is about to start the game. What is her probability of winning the big prize?

A tree diagram could help, but there are many ways to do this. You could even use expected value!

Important Stuff

4. **a.** As staircases get larger, what appears to happen to the proportion of the staircase that is covered in dots?
b. Estimate the number of dots in a staircase with 1,000 steps.

Be *thrifty* and refer back to your work from Set 4.

5. In a staircase with 100 steps, there are 3,044 dots. Use this information to calculate the number and proportion of dots in a staircase with 101 steps.

Good luck drawing this in the space below!

6. Write out Pascal's Triangle up to its seventh row. The seventh row is the first with the number 7 in it.

The first row of Pascal's triangle is

1 1

7. The notation $\binom{n}{k}$ gives the kth number in the nth row of Pascal's Triangle, counting from $k = 0$ to $k = n$ across a row. For example, $\binom{3}{0} = 1$ and $\binom{3}{1} = 3$.

Determine each of these *without use of formulas*.

Each number in lower rows is the sum of the two above it. The second row is

1 2 1

g. $\binom{4}{2}$

r. $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$

e. $\binom{5}{2}$

t. $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

a. $\binom{8}{4}$

8. The TPIR game "Five Price Tags" is very similar to Shell Game, except there are five questions to answer. If each question is a 50/50 guess, what is the probability of . . .

a. . . . getting all five questions right?

b. . . . getting exactly *two* of the questions right?

What is Macklemore's favorite Matt Damon movie?...

9. A set of 120 coin flips can be thought of as 60 independent two-flip sequences. Some of these sequences will have no heads, some will have 1, some will have 2.

a. If a two-flip sequence is picked at random from a real data set, give estimates for the probability that the sequence has no heads . . . one head . . . two heads.

b. For an entire 120-flip run of real heads and tails, give estimates for the *number* of two-flip sequences that have no heads . . . one head . . . two heads.

Neat Stuff

10. *Make a game* that is about one-third likely to be won. Explain clearly how the game is played, and what the winning condition is. The best games are simple to play but complex in their potential outcomes. Don't worry too much about making the winning probability exactly one-third.

One game would be "Roll a die and if it comes up 1 or 2, you win." But there are more interesting games!

11. In the real Shell Game on TPIR, players win chips with probability $p = 0.69$, not 0.5. What is the probability that players win the big prize?

. . . why, *Goodwill Hunting*, of course.

12. Generalize: what is the probability of winning the big prize if the probability of winning the small prizes is p? Work it out!

13. What's the probability that an integer picked from 1 to n is *square free* if

a. $n = 10$?
b. $n = 100$?
c. $n = 1000$?
d. $n = 10000$?
e. What is happening "in the long run" (as n grows larger without bound)?

An integer is *square free* if it has no square factors greater than 1. So 4 isn't a factor, 9 isn't, and so on. 30 is square free; 60 isn't. While it is possible to do this problem without technology, it's probably a good idea here.

14. This is F_5, the *Farey sequence* of order 5:

$$\left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$

F_5 is all fractions from 0 to 1, inclusive, with denominators less than or equal to 5, and is written in increasing order with all fractions in lowest terms.

a. Write out F_6, F_7, and F_8.
b. How many elements are there in F_{10}? In F_{11}?

In a previous course we called this the *Godmother sequence*, har de har. These sequences get really long, almost as long as the *Godfather sequence*.

15. Picture a long string of random digits. Longer! Start anywhere in the string and count how long it takes before every digit is seen at least once.

a. What is the minimum number of meals, um, digits, it will take before each digit is seen at least once, and what is the probability of this occurring?
b. What is the maximum number of digits it may take before this happens?
c. Say the ninth of the 10 digits just popped up for the first time. On average, how many more digits should it take until the tenth digit shows up?
d. On average, how long should it take for all ten digits to appear from the start?

This problem is sponsored by next weekend's concert hit, *Spain. The Woman.*

16. How many digits does it take before all ten of the numbers from 0-9 appear in the decimal expansion of π

(starting from the beginning)? How does the answer compare to the result from Problem 15?

17. On the 80s game show staple "Super Password", teams competed to reach 500 points before one another. Round n was worth $100n$ points.

Macklemore's favorite clue for the word *awesome* would not have been allowed.

a. List all the ways that a team could win the game. Careful: if the other team hits 500 first, that's a loss.

b. Dave calls Round 1 of this game "The Round That Scoring Forgot". Why??

Tough Stuff

18. Paul is standing at the edge of a pool, and he's a little disoriented. Each step he takes, he has a $\frac{1}{3}$ chance of stepping toward the pool, and a $\frac{2}{3}$ chance of stepping away from the pool.

Paul's disoriented to the point where he can reason neither abstractly nor quantitatively.

What is the exact probability that he will eventually fall into the pool? Note that this probability will be more than $\frac{1}{3}$ – he could enter the pool on the very first step, but there are other ways it can happen.

19. This is the sum of all unit fractions with prime denominators:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\cdots$$

Does this sum converge? If so, to what? If not, can you prove it?

TI-Nspire skill. Here are some explicit directions on how to expand the expression $(r+w)^4$ on the TI-Nspire.

You could also hit **ctrl** then N for a new document. Lots of the typical command letters work the way they're supposed to.

- Press HOME (the house). Select "New Document".
- Select Calculator to get a blank calculator screen.
- Type out the word `expand`, then the expression you want to expand. There are two sets of parentheses, one for the `expand` function and one for the expression. Your calculator line should look like `expand((r+w)^4)`

Use the little buttons: E-X-P . . .

- Hit ENTER in the bottom right and celebrate by dressing in flannel zebra jammies.

With socks on, natch.