Problem Set 7: Race Game

Opener

In today's game the player placed four price tags next to items in hope of lining up correct answers. Each item had one correct price.

This time, the tags are placed, not popped, under penalty of law.

1. How many different ways were there for our contestant to arrange the four price tags?

2. In how many different ways could the contestant earn all four prizes? three prizes? two prizes? one prize? no prizes :(

3. What is the expected number of prizes per play of this game?

The *expected number of prizes* is the average number the show will award if this game is played repeatedly.

Important Stuff

4. How many rearrangements are there for the letters in each name?

   a. LIEM
   b. EMILY
   c. DEBORAH
   d. ANNE
   e. PHYLICIA
   f. URUGUAY
   g. KAIYOORAWONGS
   h. BOBO
   i. DEEDEE
   j. DARDAR

BOBO and DARDAR are little-known and littler-used names in the Star Wars universe. Bobo Fett died on his first bounty mission. Dardar Binks was eaten by one of those underwater things.

5. **a.** Describe a way to count the number of arrangements of any word.
   b. Test your method on the words BAMMBAMM (420 ways) and LLLLRR (15 ways).

6. There are lots of ways to flip 4 heads and 3 tails in 7 coin flips . . . and lots of ways to count them.

   Write down some of the different methods you could use to figure out the answer to this problem . . . oh, and determine the answer, too.

7. **a.** Find the values of $\binom{10}{3}$ and $\binom{10}{7}$ by using Pascal's Triangle.
   b. Write a "word" using only the letters H and T that has $\binom{10}{3}$ possible arrangements.

Oh well ah everybody's heard, about your word . . .

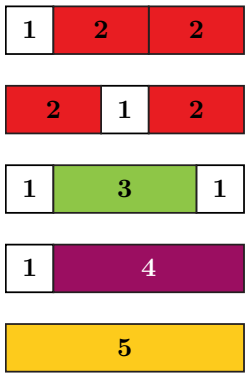
c. Write a "word" using only the letters H and T that has $\binom{10}{7}$ possible arrangements.

8. For each given n, determine the proportion of the numbers from 1 to n that have no common factor with n greater than 1.

a. $n = 9$
b. $n = 10$
c. $n = 15$
d. $n = 21$
e. $n = 35$
f. $n = 50$

So there's these rods. The rods have integer sizes. You can build "trains" that all share a common length. A "train of length 5" is a row of rods whose combined length is 5. Here are some trains of length 5:

Unless you are told otherwise, you have an unlimited supply of all the rod types, and rods can be as long as necessary. The white rod is usually called a 1, the red is a 2, and so on.



The 1-2-2 train and the 2-1-2 train contain the same rods but are listed separately, and so would 2-2-1. It's a different train if the rods are in a visibly different order. Rods of the same color are interchangeable: there is one 1-3-1 train listed, not two, but a 3-1-1 train would be different.

Macklemore saves money by looking for and making use of structures. (Due to a clerical mistake, this joke was missing from Set 5. We regret the error.)

9. **a.** How many distinct trains of length 4 can be made? Draw or make them all.
b. How many trains of length 4 are made with one rod? two rods? three? four? five? A 1-1-2 train counts as three rods, by the way.

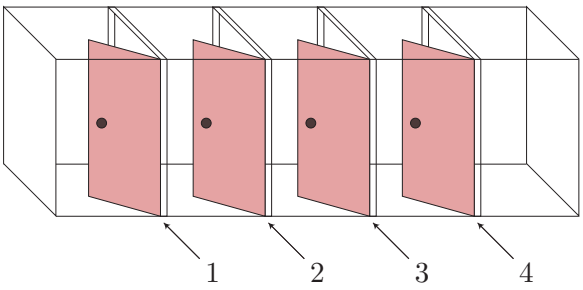
Five rods? That's a bit of a gimme.

10. Repeat Problem 9 for trains of length 5. The work you just did might be helpful . . .

Neat Stuff

11. Marty walks through the five-room corridor below from left to right. As he walks through each door, he flips a coin: heads and he keeps the door open, tails and he closes it.

The moral of this story: don't walk behind Marty in the hallway.



After walking through all four doors, he notices the pattern of open and closed doors makes a train of length 5!

- **a.** Give one example of how the coin-flipping might go and what train of length 5 is built.
- **b.** What sequence of coin flipping would result in a 2-2-1 train?
- **c.** What is the probability that Marty's flipping creates a train with one car? two cars? three cars? four cars? five cars?
- **d.** Can every possible train of length 5 be made in this way?

Marty already flipped when he built this scheme, I say. Don't mind me, I'm just constructing a viable argument and critiquing the reasoning of others. He's the one who lost his Marty-flippin' mind.

12.
- **a.** So there are 16 trains of length 5, and 16 ways to flip 4 coins. Can you think of a way to match them up?
- **b.** Using your method for matching up trains and coin flips, what train is represented by this sequence of heads and tails?

HHTHHTTHHH

There may be more than one correct answer here since it depends on how you matched up trains with coin flips.

13. Which is bigger, $\binom{120}{59}$ or $\binom{120}{61}$? *No calculators!!!*

14. At the end of Set 6 we looked briefly at a scatter plot of the number of dots against the size of staircases. Make a similar scatter plot for the data you've gathered. What sort of function would reasonably match the data?

15. Think of your original 120 coin flips as 40 independent three-flip sequences. For example, if your first 6 coin

flips are 100101, then your first two three-flip sequences are 100 and 101.

a. Start with your real flip data. Tally up the number of three-flip sequences that had no heads, one head, two heads, and three heads. (Your answers should add up to 40.)
b. Repeat for your fake coin flip data.
c. If you flip three fair coins, what is the probability that you get no heads? One head? Two heads? Three heads?
d. In 40 three-flip sequences, how many should have no heads? One head? Two heads? Three heads?
e. Go forth and test!

Two-flip sequences, then three-flip sequences, what could be next? If you can't get enough regularity from this repeated reasoning, try taking Metamucil.

16. The Race Game can be played with five prizes instead of four, with no other significant changes.

a. What's the new probability of winning all the prizes?
b. What's the new probability of not winning any prize at all? How does this compare to the probability of not winning any of the four prizes from the original game? What is the expected number of prizes per play of this game?

There are some horrible, horrible names for such a game. Please don't think about this.

17. The actual Plinko board is 12 rows tall, has "walls", and the player can drop their chip from any of 9 different places (above any of the scoring slots). See what you can do to come up with the actual expected value of the game, or the actual probability of having a chip fall into the \$10,000 slot.

Plinko is a metaphor for life: you make a choice, you get bumped around a little, and maybe you win \$10,000. Wait, what?

Tough Stuff

18. Generalize the results from Problem 16. If there are n possible prizes, what is the expected number of prizes won in the Race Game? If there are n possible prizes, what is the standard deviation of the the number of prizes won?

19. If you laid out all 2^{n-1} trains of length n, how many rods of each length from 1 to n would you need? What is the average length of cars used in making all trains of length n?

More importantly, how many *snakes* are there on these Marty-flippin' trains?