Problem Set 9: The Big Wheel . . . OF FISH!

Opener

David, Jonathan, Mariah, and Nate each spin the Wheel of Fish twice.. The Wheel is marked with the numbers 1, 2, 3, and 10. Players earn the total number of combined fish from their two spins.

Bloop bloop . bloop .. bloop .... bloop .... bloop .... bloooop .... splish.

1. What is the most likely total each person could earn from the two spins, and how likely is it?
2. What is the probability that a contestant wins at least 10 fish in their two spins?
3. What's the result when you multiply out this expression:

$$(x \qquad + x \qquad + x \qquad + x \qquad)^2$$

   Looks like you're stuck filling in the exponents, too.

Important Stuff

4. Peter plays a game with five outcomes, but only remembers a few things about its probabilities. One of the five outcomes is the most likely of all, so he calls its probability p. Two of the four other outcomes have probability $\frac{2}{3}p$, and the remaining two outcomes are even less likely, only $\frac{1}{6}p$ each.

   Those are all the possibilities. What is p?

We aren't sure but suspect Peter may have named the variable after himself.

5. Fill out four transparencies. Each is a 15-by-15 grid.
   - On a transparency, make a **blue** dot at each integer lattice point (x, y) with $1 \leqslant x, y \leqslant 15$ where the greatest common factor (GCF) of x and y is 1.
   - On another transparency, make a **red** dot at each integer point (x, y) where the GCF of x and y is 2.
   - On another transparency, make a **green** dot each integer point (x, y) where the GCF of x and y is 3.
   - Blah blah ... **black** dot ... blah blah ... GCF ... is 4. You get the idea.
   - Now pile all the transparencies together one atop another. What do you notice? Can you explain this?

Split up this work among your tablemates.

Please use transparencies! The pattern for this problem is much easier to see on transparencies than on graph paper!

Blah blah, black dot, have you any GCF? Yes sir, yes sir, 4.

6. Count the number of dots on the blue and red transparencies. Roughly how many times more blue dots are there than red dots?

7. This data comes from a 360-by-360 grid instead of a 15-by-15 grid.

Note that not all dots will be marked as blue, red, green, or black, if the greatest common factor between the coordinates is more than 4.

Color	Dots (out of 129,600)
blue	78,907
red	19,759
green	8,771
black	4,959

a. Roughly how many times more blue dots are there than red dots?
b. Roughly how many times more blue dots are there than green dots?
c. Roughly how many times more blue dots are there than black dots?

8. Check out the zone on the blue transparency with x and y between 1 and 7. See it anywhere else? Is anything like this going on for other colors? Can you use this to explain the patterns in Problem 7?

If you stare at these long enough, you might see a sailboat or a dolphin.

9. Consider a *very* large grid, colored like the ones you have been working on. Let p the probability that a point (x, y), chosen randomly, is colored blue.

a. Give an approximate value for p based on previous work in this or another problem set.
b. If the probability of picking a blue dot is p, what is the probability of picking a red dot?
c. If the probability of picking a blue dot is p, what is the probability of picking a green dot?
d. Suppose this went on forever: what is the probability, compared to p, that the pair (x, y) will have greatest common factor n?

Unlike a karaoke rendition of Thrift Shop that seemed like it went on forever, this process really *can* go on forever!

10. Let p be as in Problem 9. Use Problem 8 to explain why

$$p + \frac{1}{4}p + \frac{1}{9}p + \frac{1}{16}p + \cdots = 1$$

then use this to write an expression for the value of p.

Neat Stuff

11. **a.** Write a polynomial you could use to model rolling a fair six-sided die.

b. What's the most likely total rolled on two dice, and how likely is it? Use polynomial multiplication!

c. What's the most likely total rolled on *four* dice, and how likely is it? Use techmology!

Techmology, man. How does it work?

12. Amy rigs the Wheel of Fish so that the player will get 1, 2, 3 or 10 fish with these probabilities: $\frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{10}$.

a. Multiply out this polynomial expression:

$$\left(\frac{1}{2}x^1 + \frac{1}{5}x^2 + \frac{1}{5}x^3 + \frac{1}{10}x^{10}\right)^2$$

b. Redo problems 1 and 2 using your newfound knowledge of the rigging.

13. Esther rigs a pair of dice so that each 6 is rolled with probability 0.5 and the other five faces are rolled with probability 0.1.

They try to make her roll some fair dice, and she say no, no, no.

a. Write a polynomial you could use to model rolling this unfair six-sided die.

b. What's the most likely total rolled on four dice, and how likely is it? Uze moar techmology!

14. Build a histogram for 10 coin flips. On the horizontal is the number of heads in the 10 coin flips, and on the vertical is the probability of getting exactly that number of heads in 10 flips. Mmm, shapely.

Try to detect it. It's not too late. To whip it. Into shape.

15. Here's how Sarah paints an 11-foot wall. Starting at the left side, she paints a 1-foot section Honeydew. Then, she flips a fair coin whose sides read "run" and "switch." If she gets "run", she paints the next 1-foot section of wall using same color she just used. If she gets "switch", she switches from Honeydew to Teal (or vice versa) then uses the new color on the next 1-foot section.

Honeydew and Teal? Who would pick such crazy colors? Also, the other side of the coin should clearly say DMC.

a. If she flips RRSRSSRRRS, how will the wall be painted? How many patches of paint will there be?

By *patches* we mean one or more connected 1-foot sections of the wall that are painted the same color.

b. If Sarah ends up creating exactly 3 patches of paint on the wall, what must have been true about her coin flips?

c. What is the probability that Sarah ends up creating exactly 3 patches of paint on the wall?

Sarah is so random. (How random is she?) She's so random that when she gives away fish, she uses the Poisson distribution.

16. **a.** How many numbers less than or equal to 15 do *not* share a common factor greater than 1 with 15?

b. How many numbers less than or equal to 35 do *not* share a common factor greater than 1 with 35?

c. How many numbers less than or equal to 91 do *not* share a common factor greater than 1 with 91?

d. Multiply this out:

$$\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)$$

17. Consider the numbers 1 through 77.

Okay, we'll consider it. I guess.

a. What fraction of these numbers are divisible by 7? by 11?

b. What fraction of these numbers are *not* divisible by 7? by 11?

c. Write down the numbers 1 to 77. Cross out any number that is divisible by 7. What fraction of the original 77 numbers remain?

d. Now cross out any number that is divisible by 11. What fraction of the numbers that survived part (c) also survived this second cut?

e. What fraction of the original 77 numbers survived *both* cuts?

Now here's some numbers, they're crossed out maybe. Before you came to PCMI we missed you so, so bad.

This problem makes me thirsty for a frosty beverage for some reason.

18. Repeat problem 17 using 105 instead of 77. What changes? What fraction of the original 105 numbers survive *all three* cuts?

The first cut is the deepest, baby.

19. Consider the 36 ordered pairs of points (x, y) with x and y between 1 and 6. Look for and express some regularity in yo' repeated reasoninz.

a. What fraction of the 36 ordered pairs have *both* numbers divisible by 2?

b. What fraction of the 36 ordered pairs do *not* have *both* numbers divisible by 2?

c. What fraction of the 36 ordered pairs have *both* numbers divisible by 3?

d. What fraction of the 36 ordered pairs do *not* have *both* numbers divisible by 3?

These ordered pairs are totally krossed out! Wiggida wack.

e. Cross out any of the 36 ordered pairs where both numbers are divisible by 2. What fraction of the original 36 ordered pairs remain?

f. Cross out any remaining ordered pairs where both numbers are divisible by 3. What fraction of the numbers that survived part (e) also survived this second cut?

g. What fraction of the original 36 ordered pairs survived both cuts?

h. Generalize to find an expression for the probability that any ordered pair (x, y), with no limit on x or y, has no common factor greater than 1.

Tough Stuff

20. **a.** Let $p(n, k)$ be the probability of flipping getting a run of heads of length k (or higher) when flipping a fair coin n times. Derive a recursive definition for $p(n, k)$ based on this observation: either the run of k heads occurs at the beginning, or it doesn't. If it doesn't, there must be a tail at flip $j \leqslant k$, and the probability that the sequence of coin flips after j will contain a run of heads of length k (or higher) is $p(n - j, k)$.

Feeling lucky, $p(n, k)$?

Verily I sayeth unto thee, $p(n, k) = 0$ if $k > n$.

Sometimes you feel like a run, sometimes you don't. I usually don't.

b. Calculate the minimum number of coin flips required before the probability of getting 10 heads in a row exceeds 50%.

c. Now calculate the probability of flipping a fair coin n times and getting a longest run of length k (which may occur one or more times).

21. Develop strategies for the *continuous* version of the Big Wheel, where any number from 0 to 100 is equally likely. Rules and play are as before: a player may stop after one spin, and busts if they go over 100 total. This one might need a little calculus . . .

And not just the "you plus me equals us" kind . . .

Print this grid on transparencies for Problem 5.