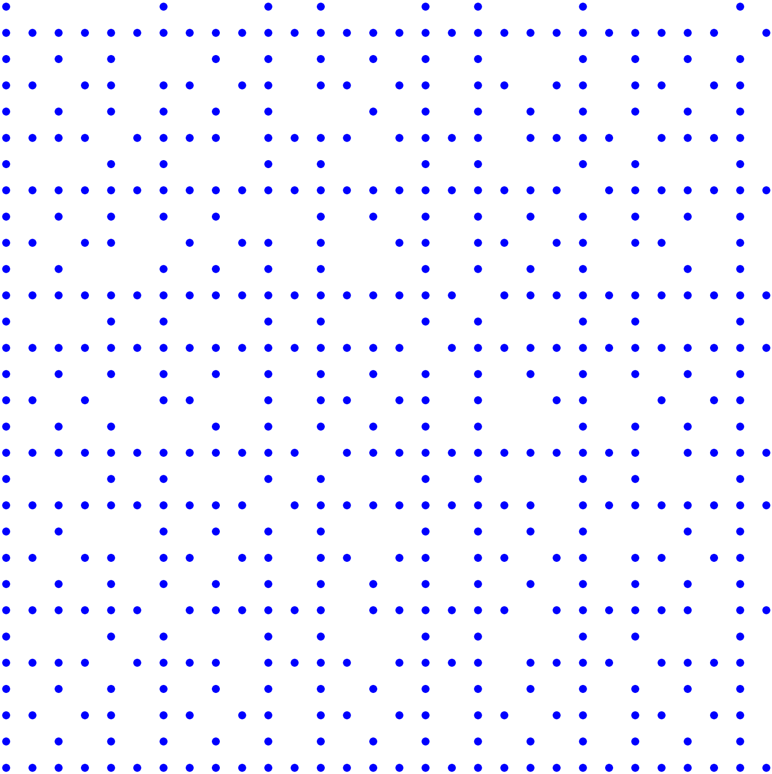
Day 9 (July 12, 2013)

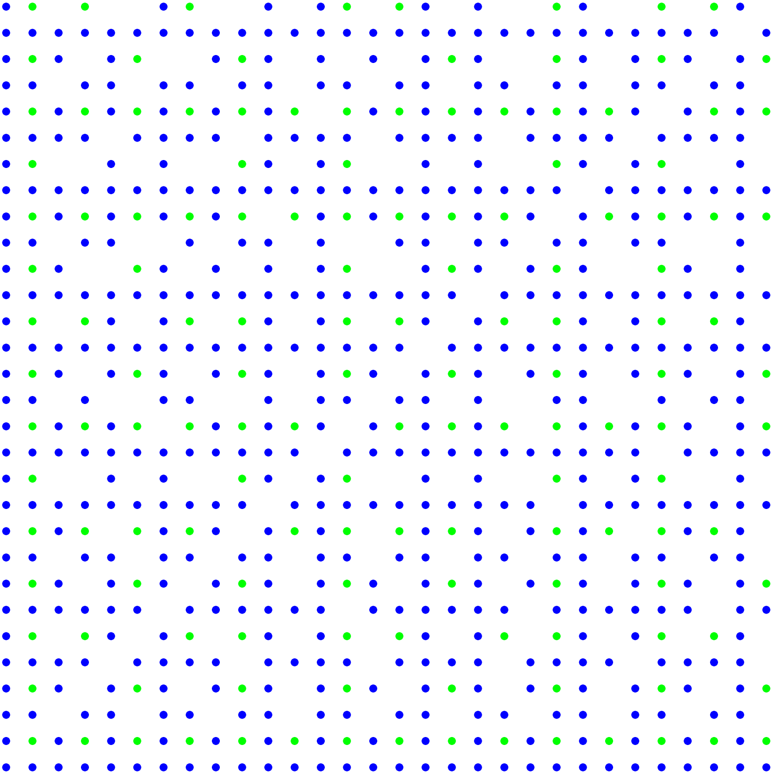
Pick two positive integers at random.
What is the probability that their GCF is 1?

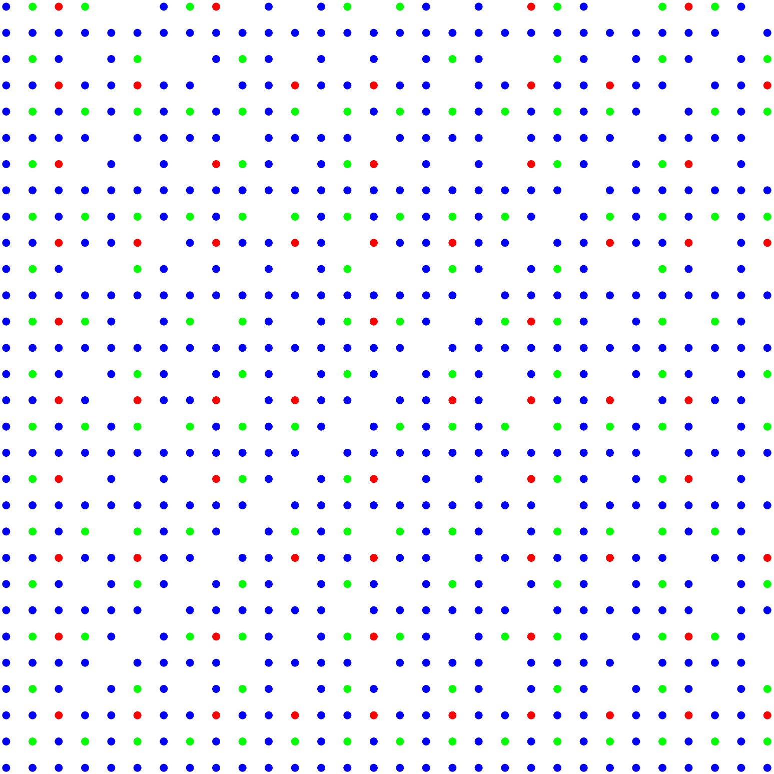
In other words, what is the probability that you pick a blue dot (dot with GCF=1) when you pick a random dot in the grid?

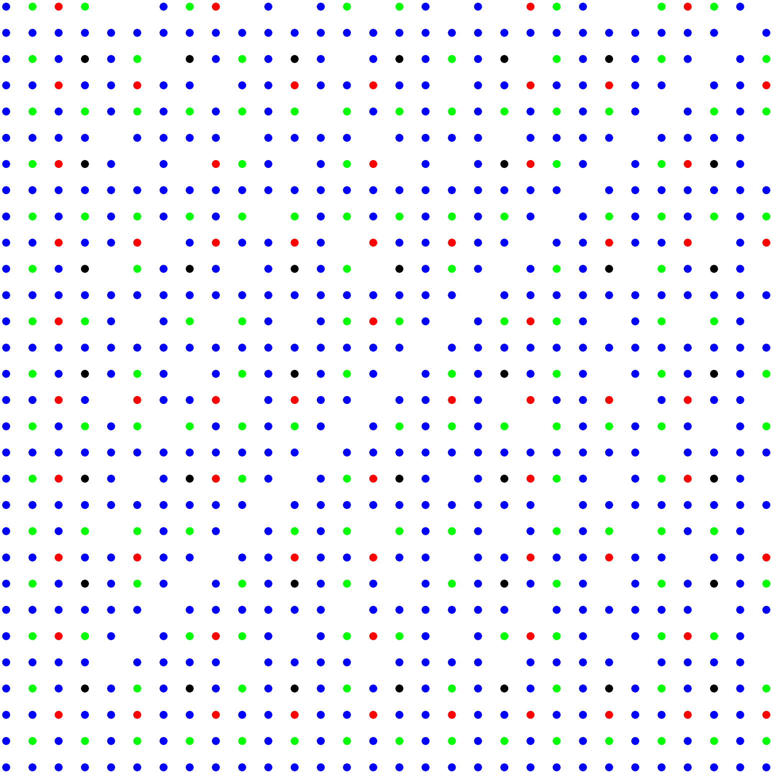
We observe three important things about the dots:

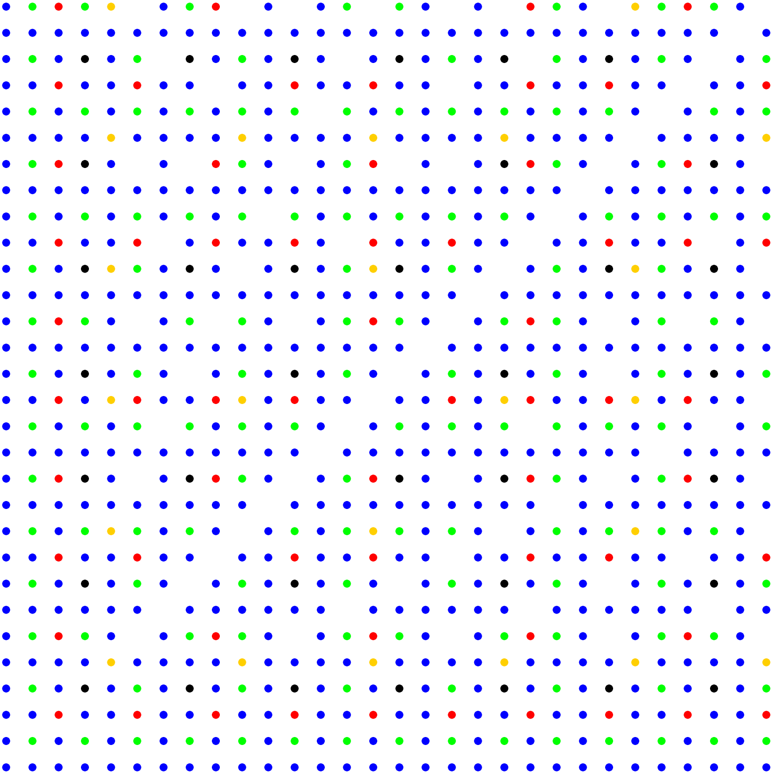
1. Every dot has a color (since every pair of positive integers has a GCF)
2. Every dot has only one color (since the GCF of two positive integers is unique)
3. The dot pattern for GCF=n is the identical to the pattern for GCF=1, just scaled by $1/n^2$.

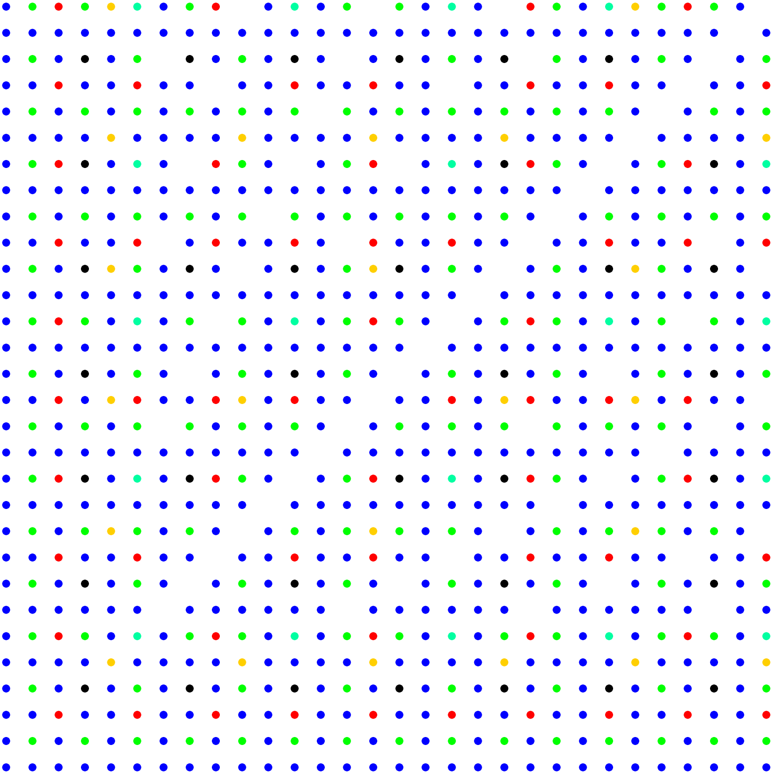


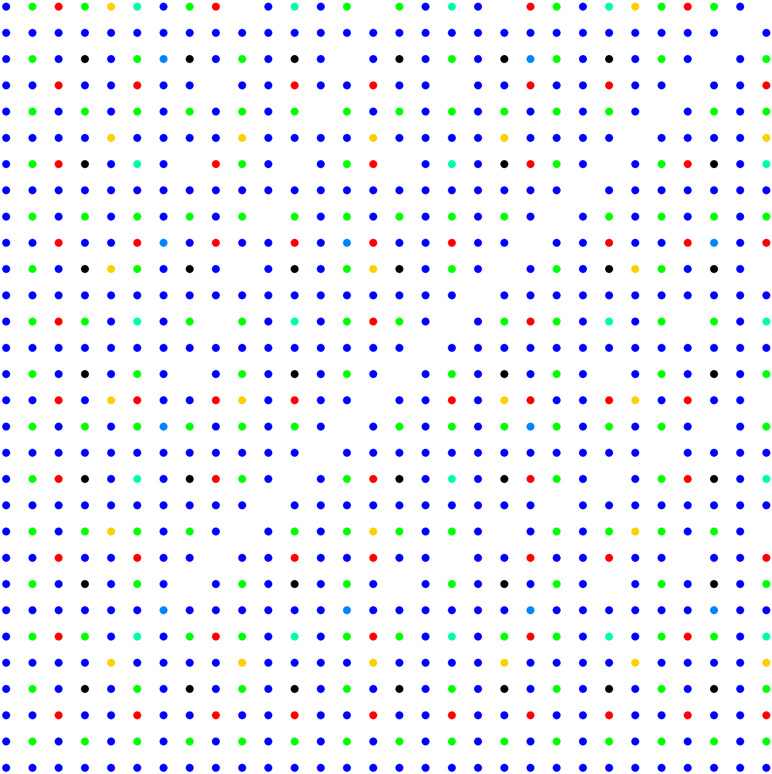


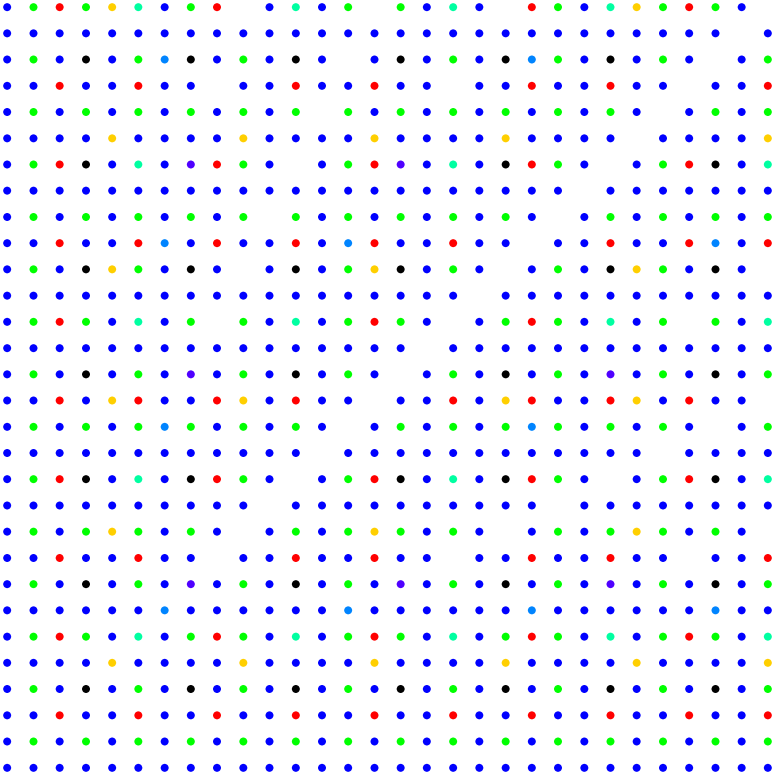


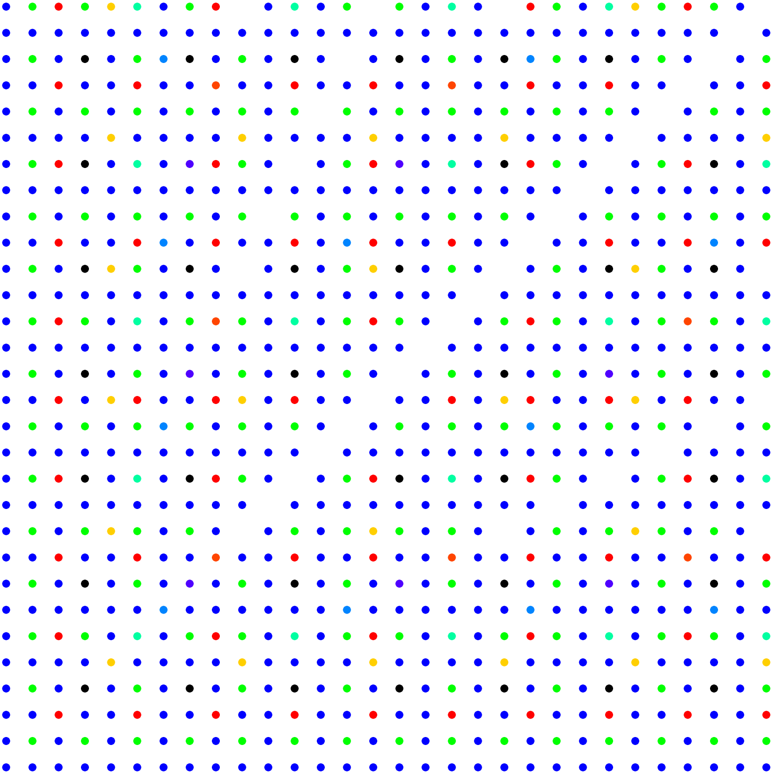


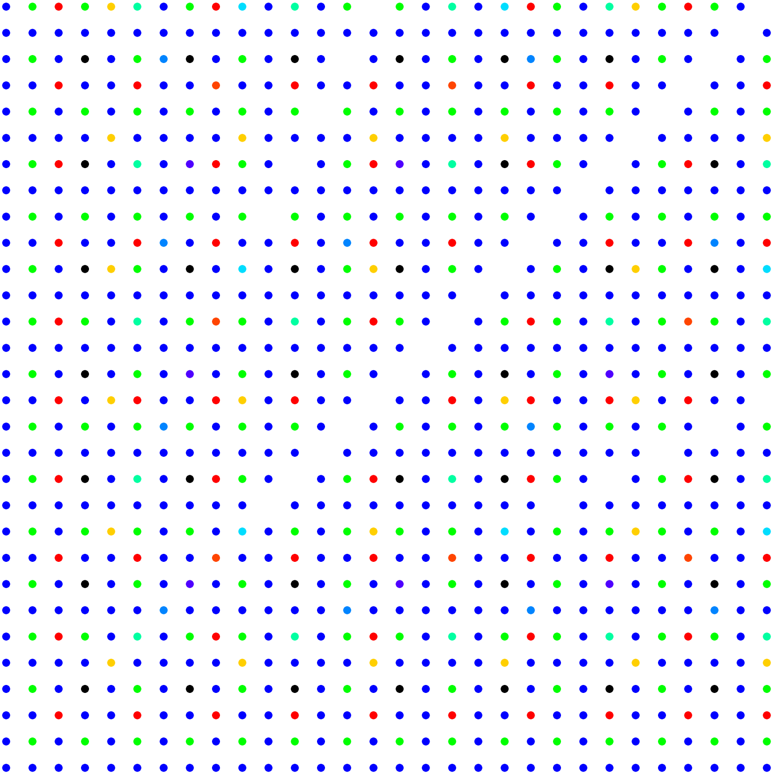


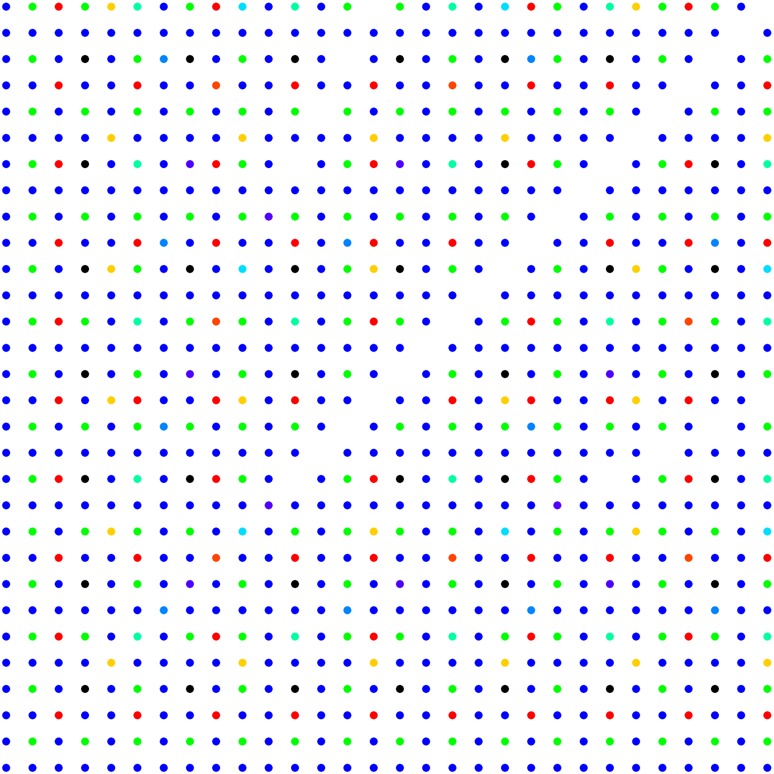


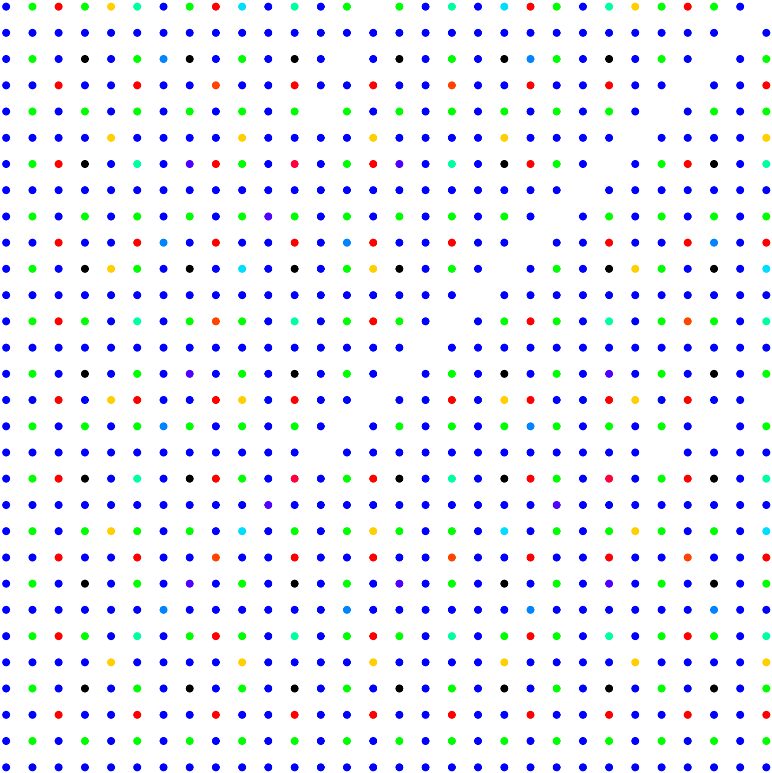


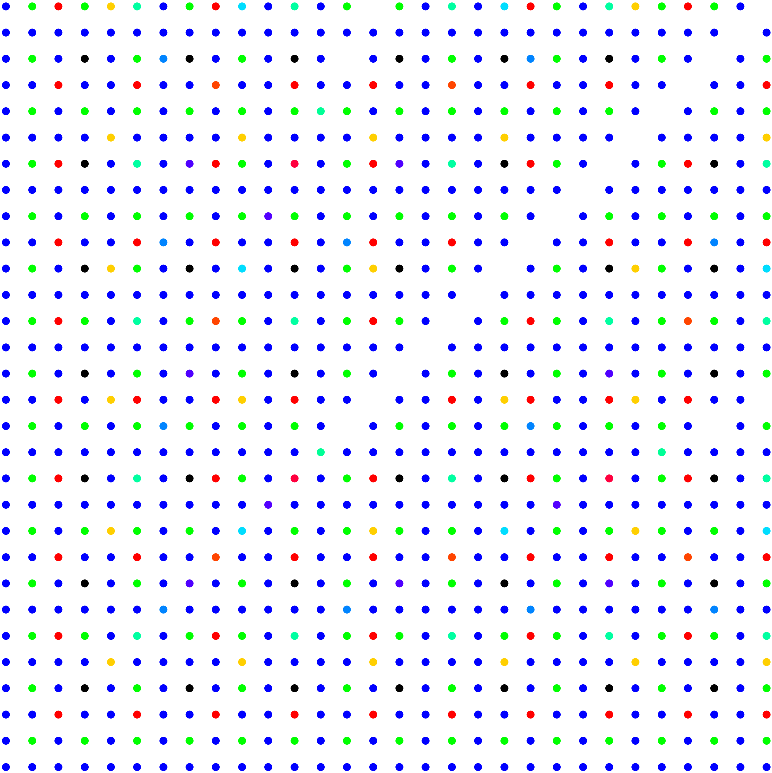


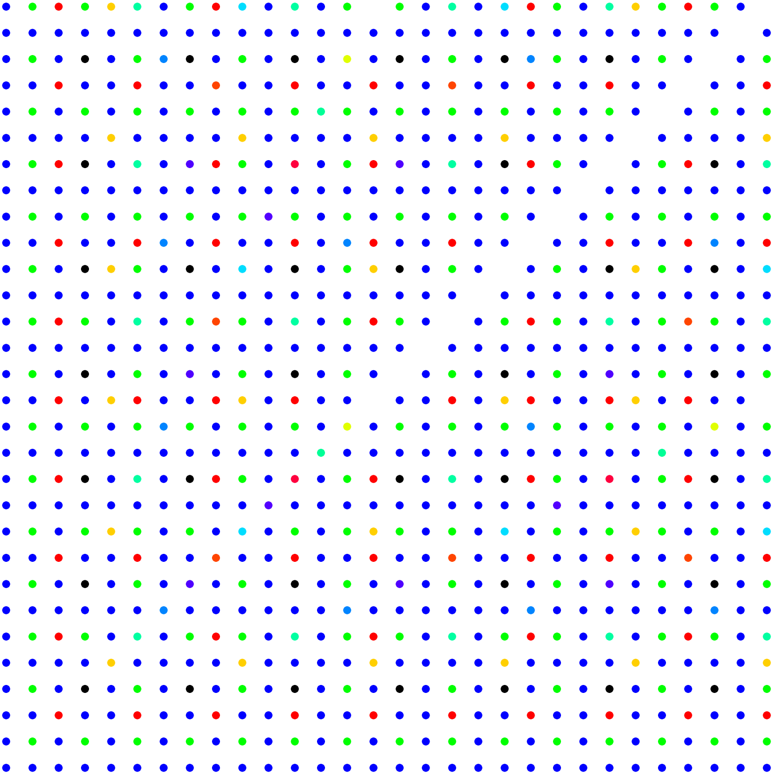


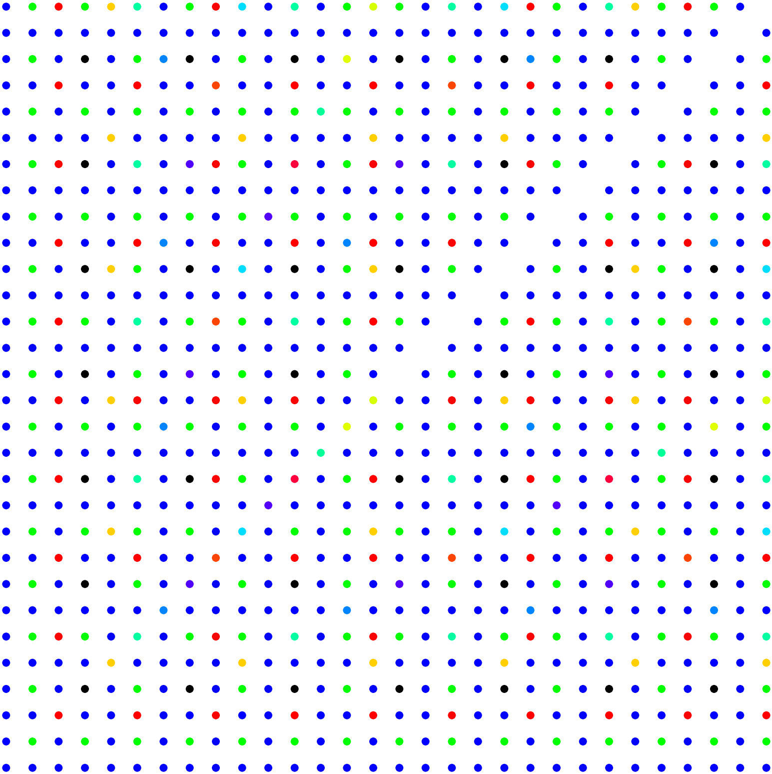


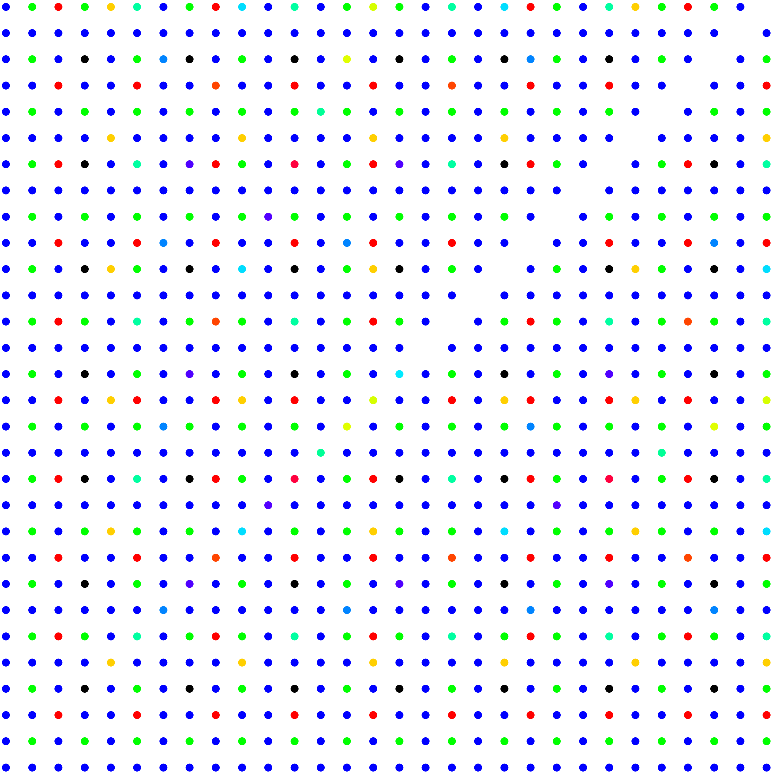


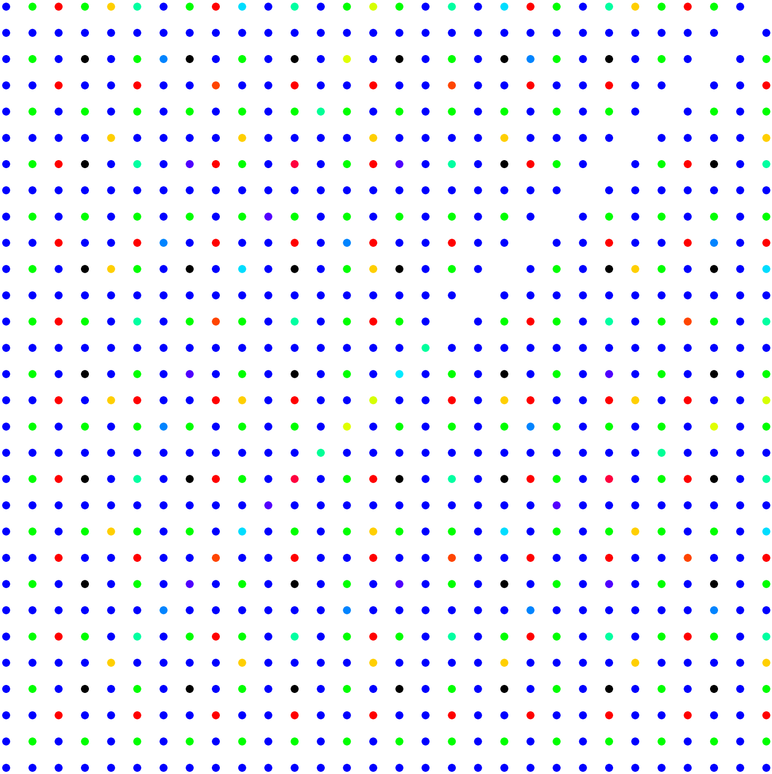


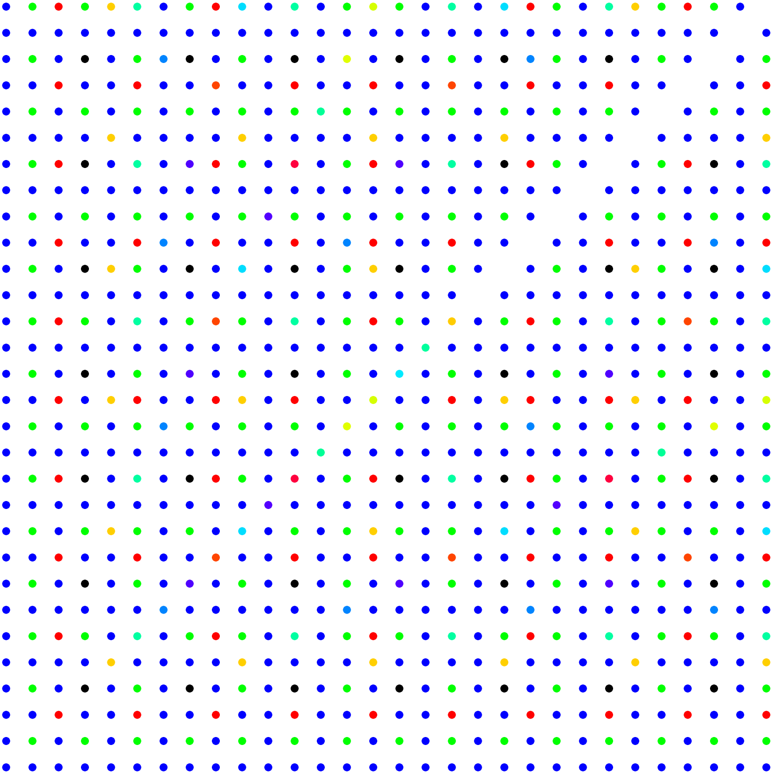


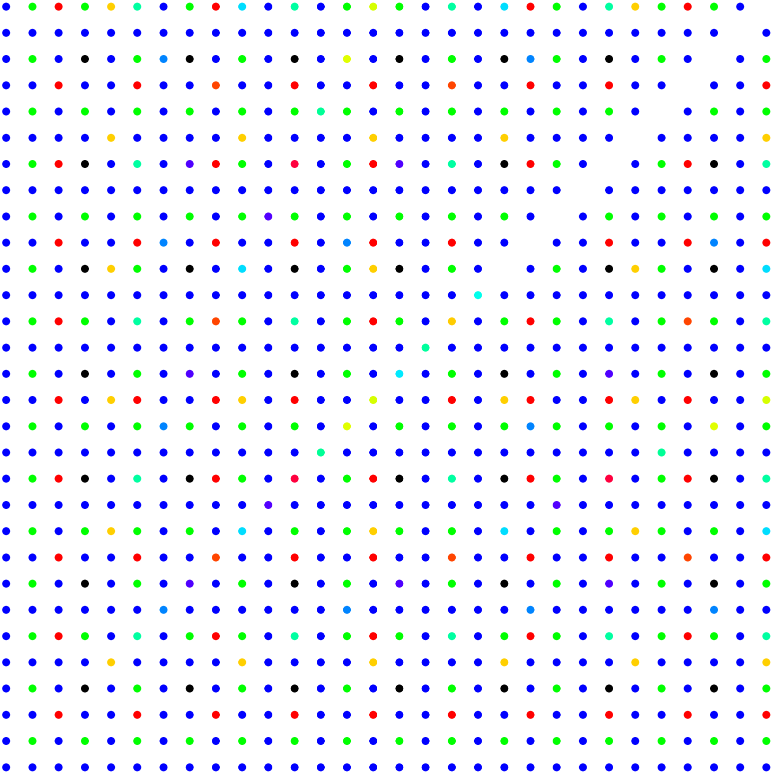


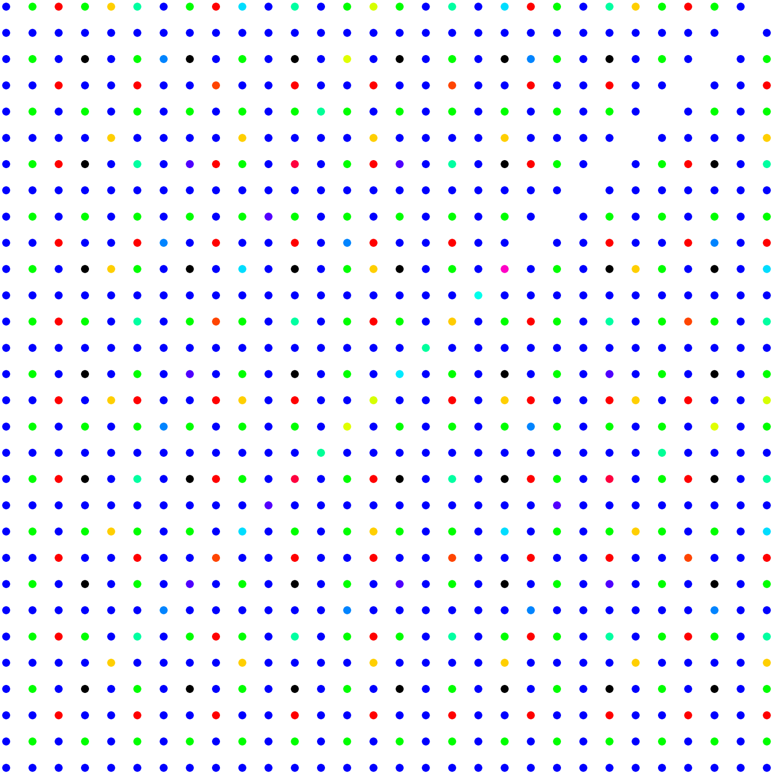


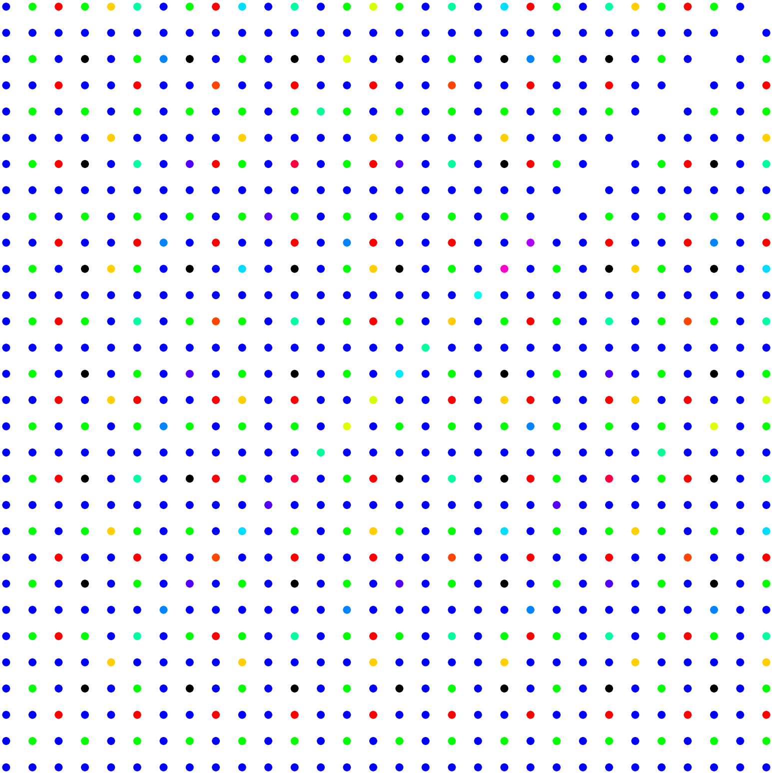


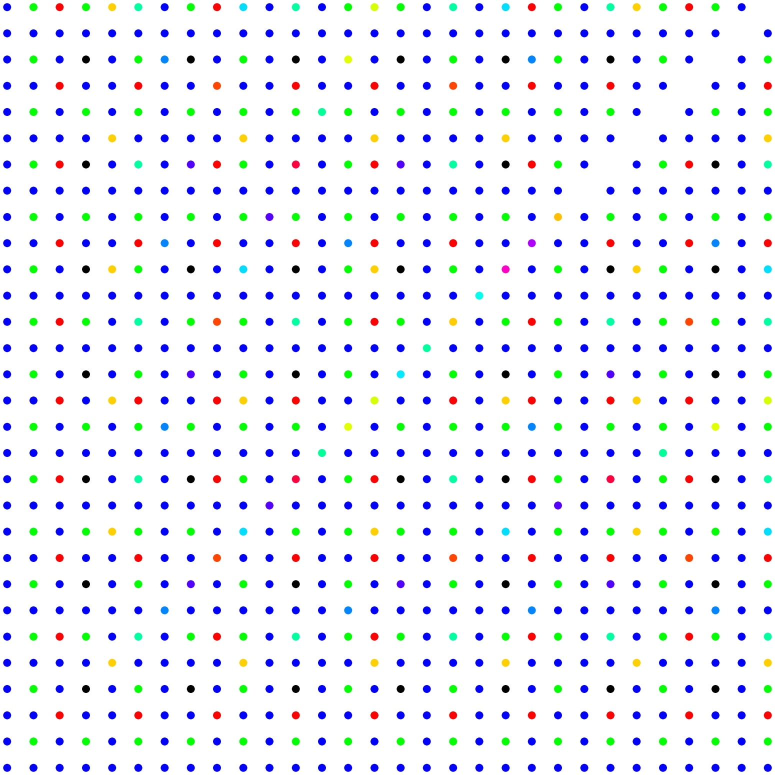


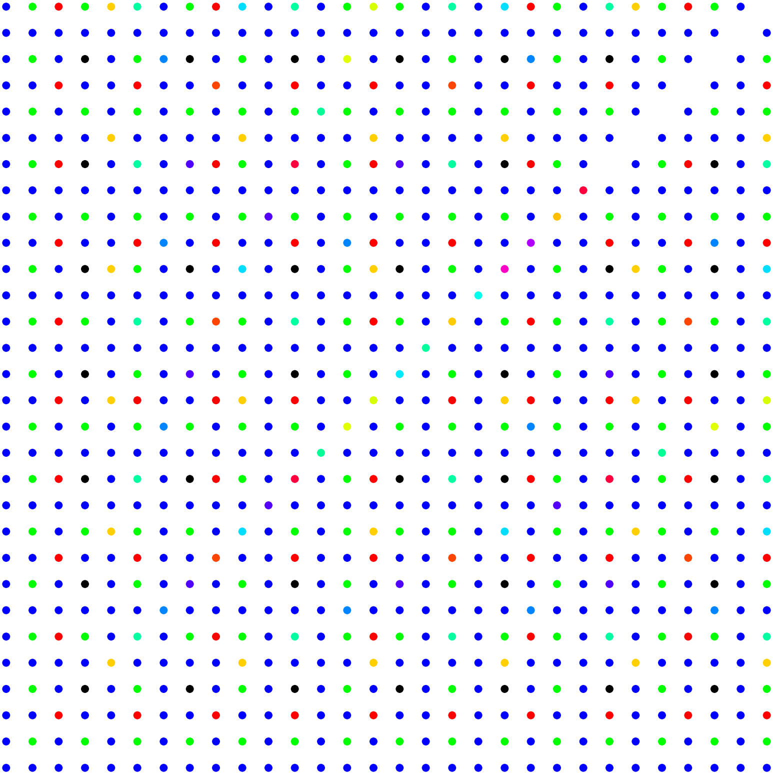


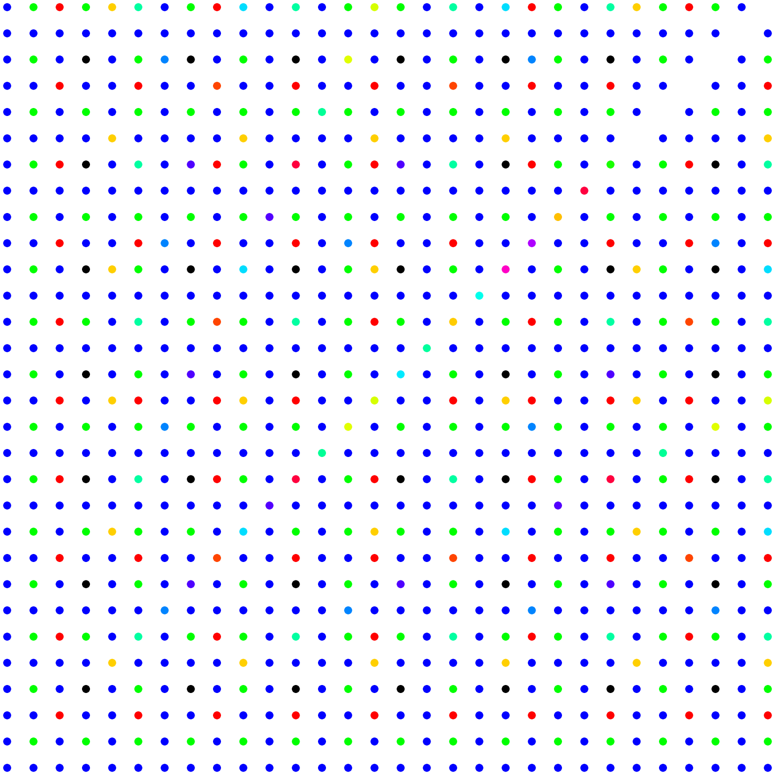


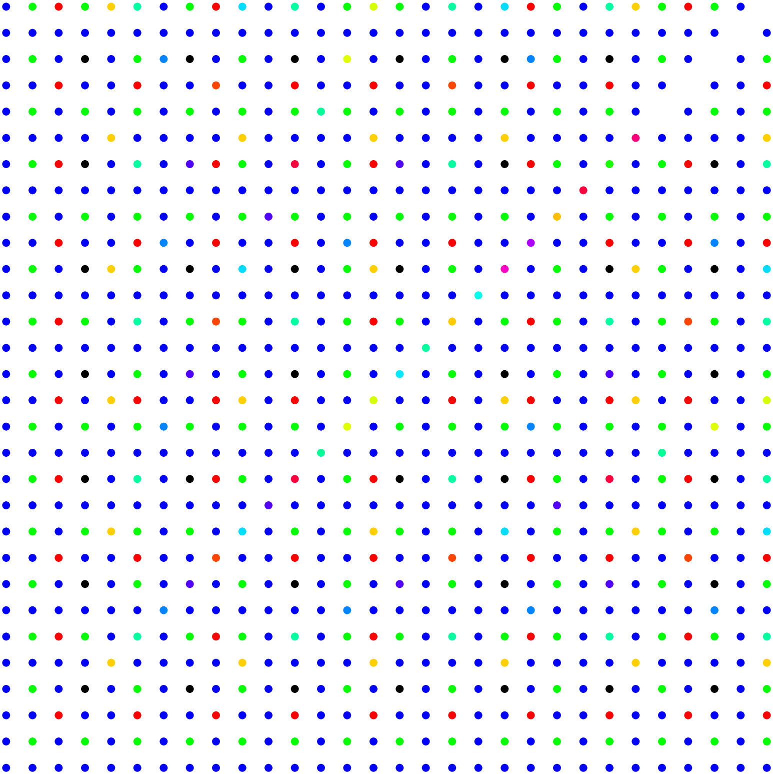


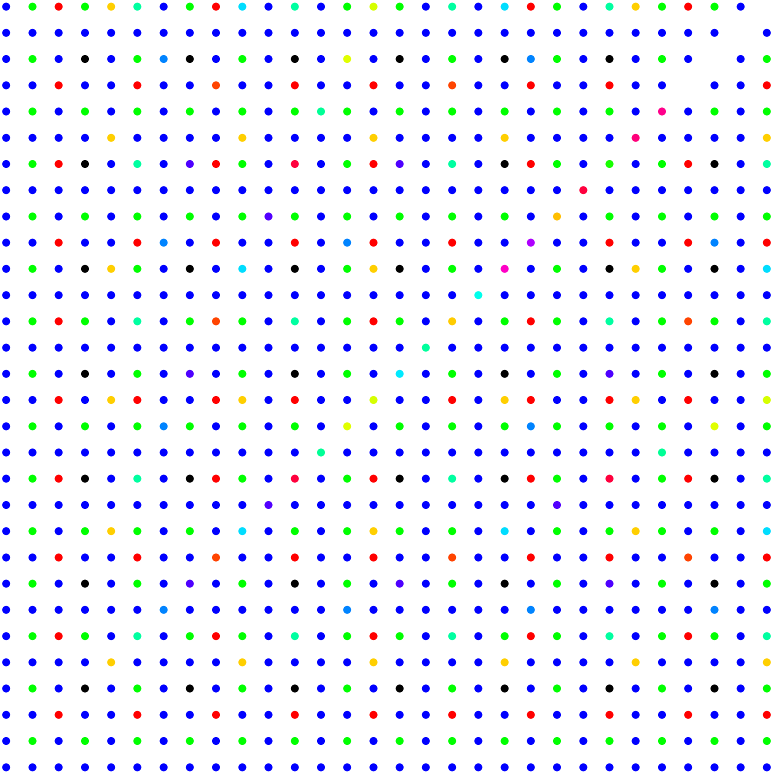


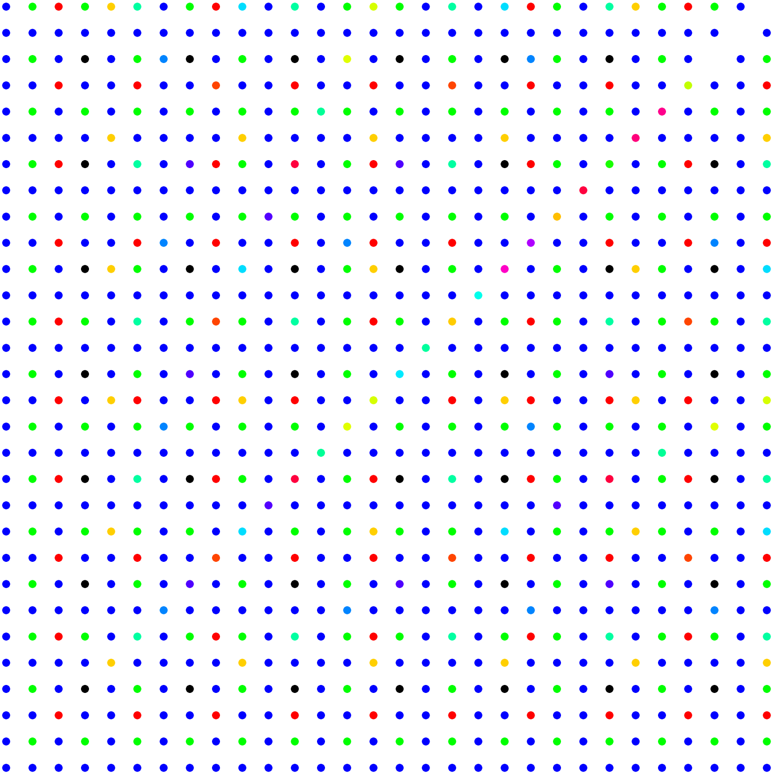


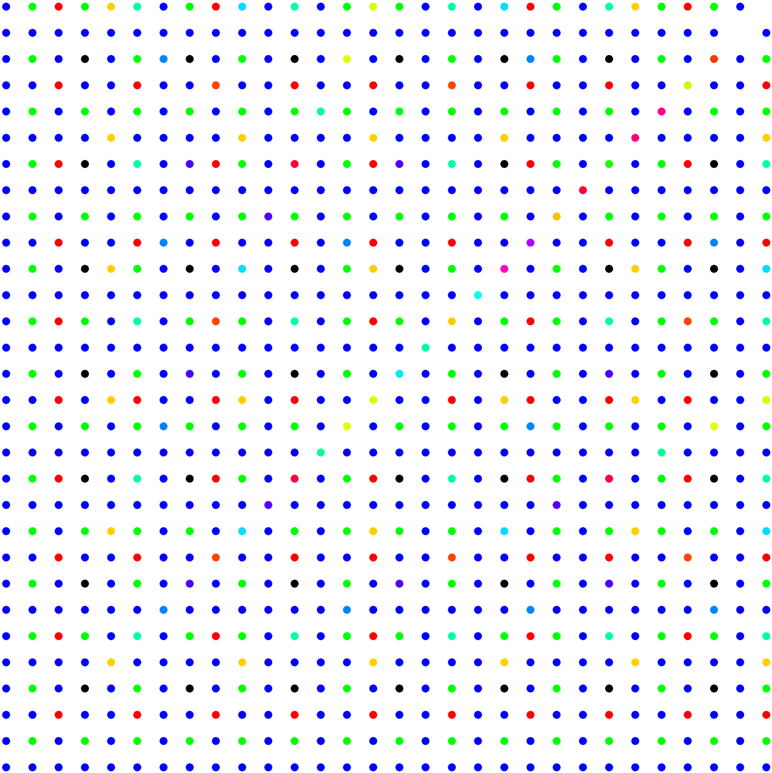


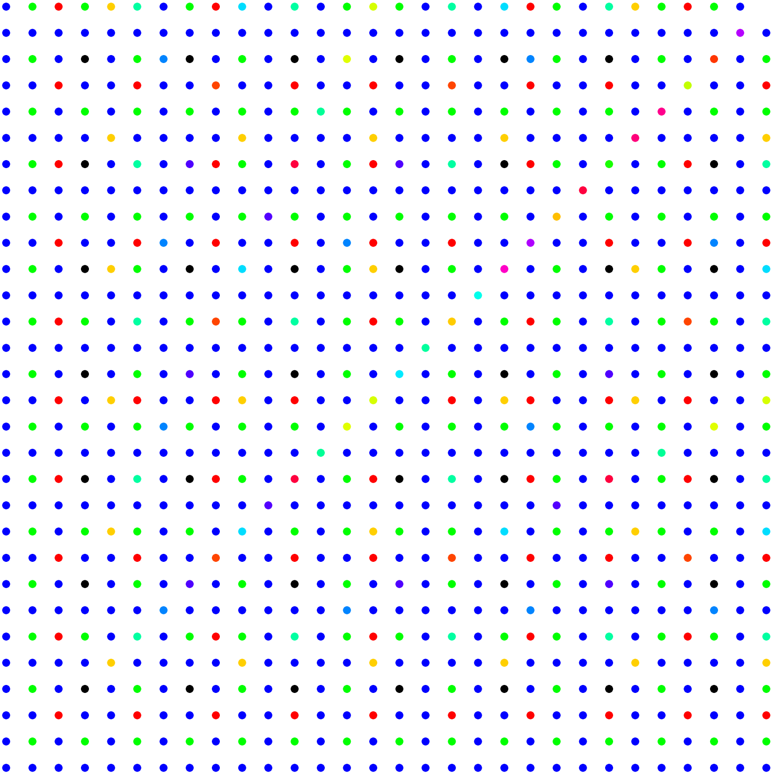


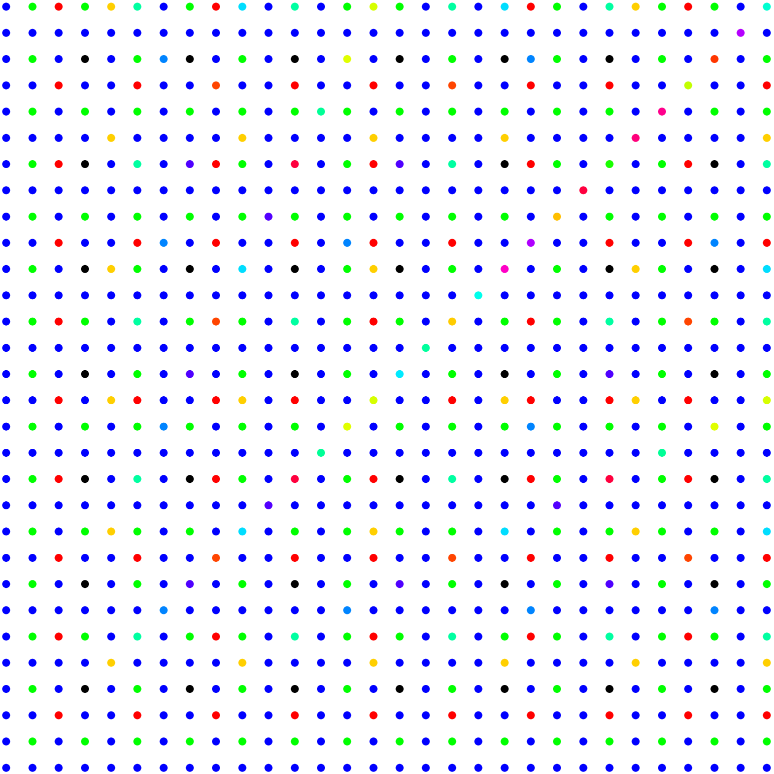


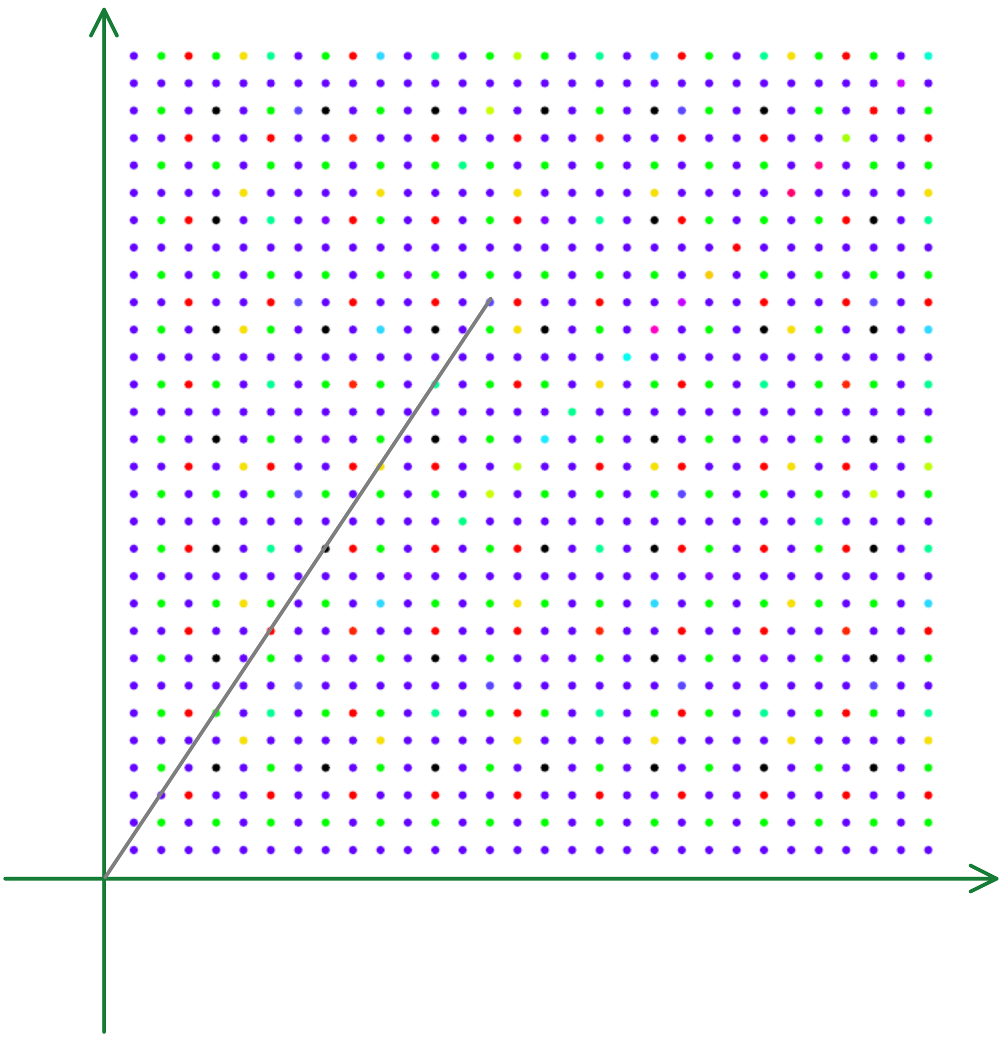












<u>Bryce's Observation:</u>

Draw a line from the origin to point (2,3). You first get a dot where GCD=1, then at (4,6) you get a dot where GCD=2, etc. You see all the dots in different colors, in order of GCF!

Let p = probability of picking 2 positive integers that have GCD = 1.

$$p + \frac{p}{4} + \frac{p}{9} + \frac{p}{16} + \cdots = 1$$

$$p\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right) = 1$$

$$p = \frac{1}{1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots}$$

But what is this infinite sum?

"Basel Problem" (first solved by Leonhard Euler in 1735)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

Therefore, $p = \frac{6}{\pi^2} \approx 0.607927\ldots$