Problem Set 11: Everybody Dice Now!

Opener

Today's game consisted of four choices made by players, then a random roll of a 20-sided die. The winner of the die rolls earned points:

A 20-sided die with the numbers 1 to 20 on it is called a *d20*. It is mostly used by nerds. So, congratulations?

- Ellie picked between *odd* and *even* for 10 points.

- Heather picked between *odd name length* and *even name length* for 20 points.

- Ellie picked between *prime* and *nonprime* for 30 points.

- Heather picked between *square* and *nonsquare* for 50 points.

1. Did Heather make a good choice between odd name length and even name length?

2. After the first two rounds, what are the possible scores, and how likely is each score?

3. After the first three rounds, what are the possible scores, and how likely is each score? Keep in mind what Ellie chose in the third round, since the probability of a number being prime isn't 0.5.

A *prime number* must have exactly two factors, so 1 is *not* prime.

4. Try multiplying this out (either by hand or through techmology) and see what happens:

$$(0.5e^{10} + 0.5h^{10})(0.5e^{20} + 0.5h^{20})(0.6e^{30} + 0.4h^{30})$$

5. Determine the probability that Ellie wins the entire game by earning more points than Heather.

Important Stuff

6. What is the probability of rolling two standard six-sided dice and . . .

Roll a *d6*, roll a d6. Na na na na na na . . .

   a. . . . getting a combined total $\geqslant 7$?

   b. . . . getting a combined total $\geqslant 7$ given that the first die was a 2?

   c. . . . getting a combined total $\geqslant 7$ given that the first die was *not* a 2?

Platonic Solids for *Everybody Dice Now* provided by C&C Plastic Factory.

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7. Repeat problem 6, except this time the goal is to get a combined total of *exactly* 7 instead of a total of *at least* 7. What changes?

8. Now imagine rolling two identical dice whose six sides are numbered 1, 2, 3, 4, 4, 4. What is the probability of . . .

   a. . . . getting a combined total of exactly 5?

   b. . . . getting a combined total of exactly 5 given that the first die was a 3 or 4?

   c. . . . getting a combined total of exactly 5 given that the first die was a 1 or 2?

9. In Set 10's game, Blue has between 2 and 14 coins, and Red rolls a six-sided die. Red wins if their six-sided roll is at least as many as the number of heads thrown by Blue. Suppose Red rolls a 2.

Blue is the table leader of Table 12. Red is the table leader of the evil Table 13, a crochety man who also played Boddicker in *Robocop*.

   a. Determine the probability that Red wins if Blue has 2 coins. 3 coins. 4 coins. 5 coins. Hope you're lookin' for patterns.

   b. 6 coins. 7 coins. Etc!

Red keeps getting mad at his table, especially his son Eric and a guy named Bobby who was asked if he could fly.

   c. Suppose it's equally likely for Blue to have any number of coins between 2 and 14, inclusive. Determine the probability that Red wins the game. Then determine the probability that Blue wins the game.

10. While he's here at PCMI, Will has asked a friend to take care of his prize-winning plant at home. Unfortunately, his friend is a bit flaky: there is a 30% chance that the friend won't show up to take care of the plant. If his friend shows up, the plant has a 90% chance of surviving. If his friend doesn't show up, the plant has only a 40% chance of surviving.
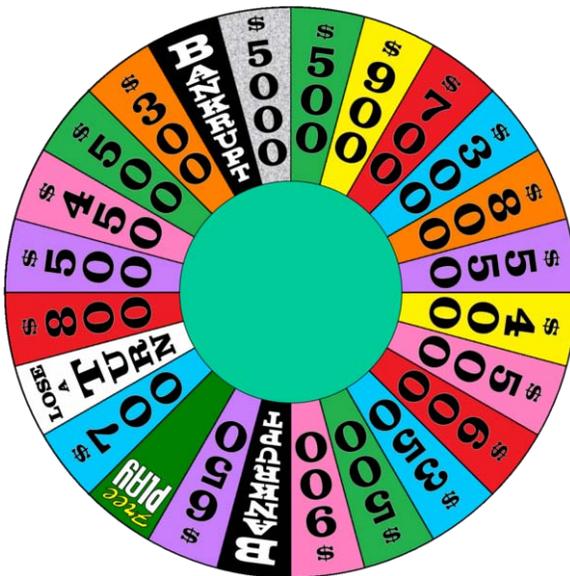
See, there are things we don't know about Will. In this case, this is probably something even Will doesn't know about himself.

   a. What is the probability that the plant survives?

   b. If Will returns home to find the plant is dead, what is the probability that his friend flaked out?

   c. If Will's friend flaked out, what is the probability that Will returns home to find the plant is dead?

11. A woman in her 40s has roughly a 1% chance of having breast cancer. A mammogram tests for breast cancer. If a woman has breast cancer, the test is positive 90% of the time and negative 10% of the time. If a woman does not have breast cancer, the test is positive 10% of the time and negative 90% of the time.

    a. If 1,000 women in their 40s are tested, estimate the number in each of four possible categories.

    b. Of those with cancer, what proportion test positive?

    c. Of *those who test positive*, what proportion have cancer?

> If you didn't get a chance to discuss this yesterday, please do so today! Several people asked yesterday whether these numbers are accurate, and they are very close to real data. The actual "false-negative" rate is closer to 8% and the actual "false-positive" rate is around 9.8%. For more information, see the WSJ's article, "Should All Women Over 40 Get Annual Mammograms?" among other resources.

Neat Stuff

12. Hey, it's the Wheel of Fortune!

http://en.wikipedia.org/wiki/File:WheelofFortuneSeason30-Round4.png

> Wheel of Fortune has now been on TV for 38 consecutive years, with Pat Sajak and Vanna White hosting since 1983. The last three Wheel of Fortune pinball machines ever produced went to Pat Sajak, Vanna White, and Bowen Kerins.

    a. What is the probability of spinning a Bankrupt?

    b. Teri is able to spin the wheel exactly once around, give or take 2 wheel positions, with equal probability. What is the probability that she spins a Bankrupt on a single spin?

    c. *Given that Teri has just spun $550*, what is the probability that her next spin will be a Bankrupt? Her next two spins? Her next three spins?

    d. *Given that Teri has just spun $5000*, what is the probability that her next spin will be a Bankrupt?

> On *Animaniacs* they would spin the Wheel of Morality to find out the show's lesson of the day. One day they spun and declared the show morally Bankrupt . . .

> Why do you suppose they put the $5000 space next to the Bankrupt?

13. You're a bank and you own 100 mortgages from 100 people. Each mortgage has the potential to earn you about $100,000 over 30 years, as long as the homeowner doesn't default. If they default, you lose $100,000.

Hey, you're a bank, now you can charge fees for stuff! Look up these fun acronyms to learn more: MBS, CDO, CDS.

 a. Suppose that each homeowner has an independent 2% chance of defaulting. What is the probability that 98 of the 100 homeowners do not default and how much will your bank earn in this case?
 b. What is your expected net profit after 30 years?
 c. If the probability of defaulting is 2%, what is the probability that the bank loses money overall on the 100 mortgages after 30 years?

This one will require tech, and perhaps some use of a giant "sigma".

 d. During the subprime mortgage boom, banks wrote mortgages for people with poor credit. Suppose each subprime borrower has an independent 40% chance of defaulting. Compute the probability that the bank will lose money on the 100 mortgages.

Of course, you did no such things. You're one of those good banks.

 e. So this seems like a slam dunk, right? Right? What mathematical mistake do you think was made?

Perhaps the mistake was calling yourself a bank!

14. A function ϕ takes a positive integer n as input. Start with n, then for any prime p that is a factor of n, multiply through by $(1 - \frac{1}{p})$. For example, consider 35. The prime factors of 35 are 5 and 7, so the result is

This is pronounced "fee" of n, not "fie". Because you're a bank, now you can charge fees for stuff.

$$\phi(35) = 35 \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right)$$

 a. Calculate $\phi(15), \phi(35)$, and $\phi(91)$.
 b. Calculate $\phi(105)$ and $\phi(231)$.
 c. Calculate $\phi(9)$ and $\phi(27)$. Be careful.
 d. Calculate $\phi(8675309)$.
 e. Describe what ϕ measures in your own words.

Factorin' is important, but big numbers can be factored through techmology.

Tough Stuff

15. What's the sum $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \cdots$ equal? You can do it using that other sum we found in Set 9.

16. $2^4 = 4^2$, and this is the only pair of positive integers x and y with $x^y = y^x$ and $x \neq y$. When else does $x^y = y^x$ when $x \neq y$ for positive reals x and y? Never? Sometimes? Try sketching a graph.

Believe it or not, this problem connects to the "split 25" problem from @ddmeyer.