# Problem Set 12: Live Election Coverage!

---

## Opener

Blue and Red play a game at 13 tables. At each table, Blue has a different number of coins (from 2 to 14, inclusive), and Red has a standard six-sided die. Red rolls the die to set a target number. Blue then tosses all his coins. Blue wins the table if he tosses *more* heads than the target number. Red wins the table if Blue doesn't beat the target number. Each table is worth a different number of votes, from 3 to 15, as provided in today's handout. There are 117 total votes. Who will win!?

Not as fun as tossing across, but still fun.

**1.** Las Cruces has 6 coins and 15 votes. What does this two-term polynomial mean?

$$\left(\frac{43b^{15}}{128} + \frac{85r^{15}}{128}\right)$$

**2.** Build a product of polynomials that represents the results from the combined outcomes of Table 4, Table 7, Table 11, and Las Cruces. Expand it using techmology then decide whether Blue or Red would be more likely to win if these were the only four tables.

I like big polynomials, and I cannot lie.

**3.** We multiplied out a really huge polynomial. We put it on the last page of this problem set, and also here:

`http://bit.ly/hugepolynomial`

Use* this document to answer these questions:

- What is the probability that Blue wins exactly 60 votes?
- What is the sum** of the coefficients in the huge polynomial?
- What is the probability that Blue defeats Red in the election?

*You can make a copy of the spreadsheet if you have your own Google account, or you can select and copy the cells into your own spreadsheet program, or you can download the spreadsheet for use on your own computer, or you can literally write down all the numbers on a spread sheet of paper, or you can make someone else do it, or you can call me Al. You can DO it! You can feel the love tonight, and have all you can eat, but only if you think you can dance.

**Protip: In most spreadsheet programs, you can highlight a group of cells and you will see the sum in the status bar.

---

### Important Stuff

**4.** You're talking with a mother about her kids. She says, "I've got three kids. Those two are a handful." You see two boys hitting each other.

**a.** What's the probability that the third child is also a boy? You may want to make a list of all possible ways a family can have three kids, then eliminate the cases you know can't happen.

Those others are brothers, you can't deny.

**b.** She adds, "My youngest child is so much calmer than they are." Does this information change the probability that the third child is also a boy?

**5.** **a.** Write a six-term polynomial that can represent the results of rolling a standard six-sided die.

**b.** Determine the probability of rolling at least a total of 16 on four dice.

**c.** Determine the probability of rolling at least a total of 16 on four dice *given that* the first die is a 1.

These dice play the hard roll.

My Wolfram Alpha don't want none unless you got terms, hun!

**6.** **a.** Determine the probability of rolling a total of exactly 10 on three dice.

**b.** If the first die roll is a 3, determine the probability of rolling a total of 10 on three dice.

**c.** If the first die roll is less than 3, determine the probability of rolling a total of 10 on three dice.

**d.** If the first die roll is 3 or more, determine the probability of rolling a total of 10 on three dice.

When Cal walks in with an itty bitty expression and a round number in the exponent it gets EXPANDED!

## Neat Stuff

**7.** Think of your 120 coin flips as 40 three-flip sequences.

**a.** If your 120 flips are real, how many of the 40 three-flip sequences do you expect to have 0, 1, 2, 3 heads? Fill in this information in the second column of the table below.

**b.** Find some other sets of coin flips, real or fake. Tally up the number of three-flip sequences that have 0, 1, 2, 3 heads and fill in this information in the latter columns below.

**c.** Figure out a way to measure how "far" your tallies in each column are away from the theoretical distribution. Do. Moar. Tests!

What's Sir Mix-A-Lot's favorite type of coin flip?

| | Expected # | Flip Set A | Flip Set B | Flip Set C | Flip Set D |
|---|---|---|---|---|---|
| 0 heads | | | | | |
| 1 head | | | | | |
| 2 heads | | | | | |
| 3 heads | | | | | |

**8.** Haven't yet done Problem 14 from Set 11? Do it now.

**9.** Complete this table for $\phi(n)$ along with the cumulative total of all $\phi$ values from 1 to $n$.

| $n$ | $\phi(n)$ | $\sum \phi(n)$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 2 |
| 3 | 2 | 4 |
| 4 | | |
| 5 | | |
| 6 | | 12 |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | 32 |

How many dots are there in a staircase with 8 stairs? 9 stairs? Hmm...

**10.** Build a grid with the numbers 1 through 15 as labels on the sides. You now have 225 ordered pairs.

225 is a square number, but only because it doesn't get any of these references.

**a.** What fraction of the 225 ordered pairs do *not* have *both* numbers divisible by 3?

**b.** What fraction of the 225 ordered pairs do *not* have *both* numbers divisible by 5?

**c.** Cross out any of the 225 ordered pairs where both numbers are divisible by 3. What fraction of the original 225 ordered pairs remain?

**d.** Cross out any remaining ordered pairs where both numbers are divisible by 5. What fraction of the numbers that survived part (c) also survived this second cut?

Survive and advance! Survivor has now been on TV for 225 seasons.

**e.** What fraction of the original 225 ordered pairs survived both cuts?

**f.** Multiply this out:

$$\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)$$

**11.** Calculate this product as long as necessary:

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^2}\right)$$

The giant $\pi$ just means multiply, so this is the same thing as

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{11^2}\right)\cdots$$

where each denominator has a prime.

When you multiply all these "slightly less than 1" things together, what happens?

It's sigma for sum and pi for product, if that helps to remember them.

Take enough terms until you're happy. Are you happy yet? How about now? How about *now?* Come on, just get happy already.

**12.** Determine the expected number of votes Blue will earn in the election. There is more than one good way to do this, so try to find more than one path.

Double up: uh-uh!

**13.** Andrea and Laurie play a coin-flipping game. Andrea wins on heads, Laurie on tails. The first player to win four flips is the champion. What is the expected *number of coin flips* necessary to determine the champion?

Who is the champion, my friend?

**14.** Rina and Soledad play a coin-flipping game, each with their own coin, alternating turns. Both of them have rigged their coins to win for their side with probability $p = 0.6$. Again, the first player to win four flips is the champ. What is the expected number of coin flips necessary to determine the champion?

Who'll keep on fighting to the end?

**15.** **a.** That huge expanded polynomial we saw today? How could you use a derivative to find the expected numbers of votes for blue and red?

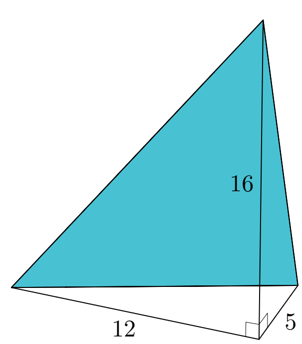
**b.** What happens if you differentiate the factored version? Product rule action: it's fan-tastic.

Bit rusty on your calculus? Just skip this silly problem!

**16.** Constance cuts a corner out of a cube, creating a tetrahedron. From the corner to the cut, the three segments have lengths 5, 16, and 12.

Say that ten times fast!

This tetrahedron is little in the middle but it's got much back.



**a.** Find the area of each face of the tetrahedron. One face's area is $2\sqrt{2929}$, but we won't tell you which!

**b.** Find the *square* of the area of each face of the tetrahedron. Notice anything interesting?

But but they're triangles, not squares!

**c.** Try another corner-cut: does it work too? Proof?

**17.** Consider ten playing cards: an ace through nine, and a joker. The cards are laid face down randomly. You grab as many cards as you want, still face down, then flip them all over at once. You win \$1 multiplied by the sum of the card values. BUT. If you flip over the joker, you win nothing. How many cards should you take to maximize your expected return?

You lose! Good day, sir! Also, we like big BUTs.

What would change if all nine non-joker cards were aces? Eight aces and a ten?

## Tough Stuff

**18.** Find a corner-cut that gives three Pythagorean triples, or prove that it can't be done. For example, the 5, 16, 12 cut from Problem 16 produces two Pythagorean triples (5-12-13 and 12-16-20) but not three.

---

| Group | Coins Used | Votes Up For Grabs | Prob Blue Wins |
|---|---|---|---|
| PC Table 1 | 5 | 3 | $\frac{49}{192} \approx 0.2552$ |
| PC Table 2 | 11 | 4 | $\frac{8905}{12288} \approx 0.7247$ |
| PC Table 3 | 12 | 5 | $\frac{19295}{24576} \approx 0.7851$ |
| PC Table 4 | 10 | 6 | $\frac{1343}{2048} \approx 0.6558$ |
| PC Table 5 | 9 | 7 | $\frac{297}{512} \approx 0.5801$ |
| PC Table 6 | 7 | 8 | $\frac{107}{256} \approx 0.4180$ |
| PC Table 7 | 8 | 9 | $\frac{1}{2} = 0.5000$ |
| PC Table 8 | 4 | 10 | $\frac{17}{96} \approx 0.1771$ |
| PC Table 9 | 3 | 11 | $\frac{5}{48} \approx 0.1042$ |
| PC Table 10 | 2 | 12 | $\frac{1}{24} \approx 0.0417$ |
| PC Table 11 | 14 | 13 | $\frac{86293}{98304} \approx 0.8778$ |
| PC Table 12 | 13 | 14 | $\frac{41099}{49152} \approx 0.8362$ |
| Las Cruces eTable | 6 | 15 | $\frac{43}{128} \approx 0.3359$ |

A really huge polynomial:

$$
\begin{aligned}
& \left(\frac{49b^3}{192}+\frac{143r^3}{192}\right)\left(\frac{8905b^4}{12288}+\frac{3383r^4}{12288}\right)\left(\frac{19295b^5}{24576}+\frac{5281r^5}{24576}\right)\left(\frac{1343b^6}{2048}+\frac{705r^6}{2048}\right) \\
\times & \left(\frac{297b^7}{512}+\frac{215r^7}{512}\right)\left(\frac{107b^8}{256}+\frac{149r^8}{256}\right)\left(\frac{b^9}{2}+\frac{r^9}{2}\right)\left(\frac{17b^{10}}{96}+\frac{79r^{10}}{96}\right)\left(\frac{5b^{11}}{48}+\frac{43r^{11}}{48}\right) \\
\times & \left(\frac{b^{12}}{24}+\frac{23r^{12}}{24}\right)\left(\frac{86293b^{13}}{98304}+\frac{12011r^{13}}{98304}\right)\left(\frac{41099b^{14}}{49152}+\frac{8053r^{14}}{49152}\right)\left(\frac{43b^{15}}{128}+\frac{85r^{15}}{128}\right) \\
= & \; 0.000002187b^{117}+0.000006384b^{114}r^{3}+0.0000008310b^{113}r^{4}+0.0000005987b^{112}r^{5} \\
+ & \; 0.000001148b^{111}r^{6}+0.000004009b^{110}r^{7}+0.000004793b^{109}r^{8}+0.000005766b^{108}r^{9} \\
+ & \; 0.00001522b^{107}r^{10}+0.00002862b^{106}r^{11}+0.00005895b^{105}r^{12}+0.00003374b^{104}r^{13} \\
+ & \; 0.00006407b^{103}r^{14}+0.0001692b^{102}r^{15}+0.00003952b^{101}r^{16}+0.00005374b^{100}r^{17} \\
+ & \; 0.0001072b^{99}r^{18}+0.0001690b^{98}r^{19}+0.0001976b^{97}r^{20}+0.0003187b^{96}r^{21}+0.0004910b^{95}r^{22} \\
+ & \; 0.0007690b^{94}r^{23}+0.0005378b^{93}r^{24}+0.0009315b^{92}r^{25}+0.001678b^{91}r^{26}+0.0007258b^{90}r^{27} \\
+ & \; 0.0008532b^{89}r^{28}+0.001375b^{88}r^{29}+0.002254b^{87}r^{30}+0.002161b^{86}r^{31}+0.002587b^{85}r^{32} \\
+ & \; 0.004877b^{84}r^{33}+0.003685b^{83}r^{34}+0.003084b^{82}r^{35}+0.008606b^{81}r^{36}+0.004355b^{80}r^{37} \\
+ & \; 0.005479b^{79}r^{38}+0.006041b^{78}r^{39}+0.009447b^{77}r^{40}+0.01139b^{76}r^{41}+0.009844b^{75}r^{42} \\
+ & \; 0.01015b^{74}r^{43}+0.01435b^{73}r^{44}+0.01502b^{72}r^{45}+0.01132b^{71}r^{46}+0.01372b^{70}r^{47}+0.01988b^{69}r^{48} \\
+ & \; 0.01879b^{68}r^{49}+0.01872b^{67}r^{50}+0.02834b^{66}r^{51}+0.01560b^{65}r^{52}+0.02231b^{64}r^{53} \\
+ & \; 0.01894b^{63}r^{54}+0.02493b^{62}r^{55}+0.02557b^{61}r^{56}+0.03025b^{60}r^{57}+0.02624b^{59}r^{58}+0.03237b^{58}r^{59} \\
+ & \; 0.03248b^{57}r^{60}+0.02011b^{56}r^{61}+0.02399b^{55}r^{62}+0.02827b^{54}r^{63}+0.03147b^{53}r^{64}+0.03168b^{52}r^{65} \\
+ & \; 0.03181b^{51}r^{66}+0.02174b^{50}r^{67}+0.03176b^{49}r^{68}+0.02103b^{48}r^{69}+0.02340b^{47}r^{70}+0.02242b^{46}r^{71} \\
+ & \; 0.02834b^{45}r^{72}+0.02071b^{44}r^{73}+0.01973b^{43}r^{74}+0.02203b^{42}r^{75}+0.01299b^{41}r^{76}+0.01442b^{40}r^{77} \\
+ & \; 0.01568b^{39}r^{78}+0.01579b^{38}r^{79}+0.01146b^{37}r^{80}+0.01418b^{36}r^{81}+0.008697b^{35}r^{82}+0.006805b^{34}r^{83} \\
+ & \; 0.006823b^{33}r^{84}+0.008396b^{32}r^{85}+0.007853b^{31}r^{86}+0.004772b^{30}r^{87}+0.004940b^{29}r^{88} \\
+ & \; 0.004462b^{28}r^{89}+0.003267b^{27}r^{90}+0.002820b^{26}r^{91}+0.002966b^{25}r^{92}+0.002591b^{24}r^{93} \\
+ & \; 0.002135b^{23}r^{94}+0.002278b^{22}r^{95}+0.0008166b^{21}r^{96}+0.0008662b^{20}r^{97}+0.0009412b^{19}r^{98} \\
+ & \; 0.001231b^{18}r^{99}+0.0006581b^{17}r^{100}+0.0003617b^{16}r^{101}+0.0004388b^{15}r^{102} \\
+ & \; 0.0002500b^{14}r^{103}+0.0002936b^{13}r^{104}+0.0001849b^{12}r^{105}+0.0001908b^{11}r^{106}+0.00009927b^{10}r^{107} \\
+ & \; 0.0001962b^{9}r^{108}+0.00003429b^{8}r^{109}+0.00003975b^{7}r^{110}+0.00003316b^{6}r^{111}+0.00006360b^{5}r^{112} \\
+ & \; 0.00004582b^{4}r^{113}+0.000005964b^{3}r^{114}+0.00001741r^{117}
\end{aligned}
$$

OMG, Becky. Look at the size of that polynomial. It looks like one of those rap guys' polynomials. But you know, who understands those rap guys anyway? They only use this polynomial because it looks like it's totally expanded. I mean, it's just so big. I can't believe... it's just so huge, it's like, out there! I mean, gross. Look! It's just so *expanded*.