Problem Set 13: Recount

Opener

Blue and Red are at it again. At each of the 13 tables, Blue has a different number of coins (from 2 to 14, inclusive). This time, Red rolls the die to set a target number *for all 13 tables*. Blue then tosses all his coins. Blue wins the table if he tosses *more* heads than the target number. Red wins the table if Blue doesn't beat the target number. Each table is worth a different number of votes, from 5 to 66, as provided at the end of today's problem set. There are 359 total votes. Who will win!?

Due to voter irregularities, this election will be overseen by noted election watchdog Jessie Pilgrim. Or maybe his daughter, Jessie's Girl.

The number of votes at each table is the rounded-off average of the electoral votes of the participants at the table. Sorry, Las Cruces! Also a *lot* of people live in Indonesia.

1. Las Cruces has 6 coins and 5 votes. What does this two-term polynomial mean?

$$\left(\frac{11b^5}{32} + \frac{21r^5}{32}\right)$$

2. We multiplied out a really huge polynomial. We put it on the last page of this problem set, and also on this spreadsheet:

   `http://bit.ly/hugepolynomial`

   The huge polynomial stops at a term marked $b^{255}r^{104}$. Why does that happen?

3. There are many gaps in the polynomial, too: there is a b^5r^{354} term, then the next one is $b^{17}r^{342}$. Why does that happen?

4. What is the sum of the coefficients in the huge polynomial? Why?

5. *Use a copy of the spreadsheet* to answer these questions about the election.
   - **a.** What is the most likely number of votes for Blue and Red, and how likely is it?
   - **b.** What is the probability that Blue defeats Red in the election?

⋆You can make a copy of the spreadsheet if you have your own Google account, or you can select and copy the cells into your own spreadsheet program, or you can download the spreadsheet for use on your own computer.

⋆⋆Protip: In most spreadsheet programs, you can highlight a group of cells and you will see the sum in the status bar.

Remember that one time at math camp where you typed in the entire monster polynomial to determine a probability? *Don't do that! Use the spreadsheet!*

Important Stuff

6. Alicia rolls three tetrahedral dice (d4), each with the numbers 1 through 4 on its faces. Find the probability that Alicia rolls a sum of 5 on three tetrahedral dice.

7. Why does multiplying polynomials help us calculate things about independent events? Write a short explanation. If possible, use these words: term, probability, exponent, coefficient, independent.

I ain't talkin' bout FOIL, 'cuz aluminum parts are made for toys! (Not really.)

8. David and Deborah play a game 7 times in a row. Deborah has a 60% chance of winning each game. Find the probability that Deborah wins more games than David in the 7-game series.

9. Diana looks back at the problem about the probability that the GCF of two numbers is 1.

Remember that one time at math camp where you learned this probability was $\frac{6}{\pi^2}$? That was awesome.

- **a.** For two numbers to have a GCF of 1, they can't both be divisible by 2. What's the probability that two integers chosen randomly are *both* divisible by 2?
- **b.** Show that the probability that the two integers *aren't* both divisible by 2 is given by $(1 - \frac{1}{4})$.
- **c.** What's the probability that two integers chosen at random are both divisible by 3? *Aren't* both divisible by 3?
- **d.** What's the probability that two integers chosen at random aren't both divisible by 2 *and* aren't both divisible by 3?
- **e.** Diana says she doesn't have to worry about 4. Why?
- **f.** What's the probability that two integers chosen at random aren't both divisible by 5?
- **g.** Diana wants to use this idea to write an expression for the probability that the two integers don't have any common factors greater than 1. How far does she need to go? What kinds of terms are there, and what do you do with them? What does the expression look like? Could this actually work?!

10. In yesterday's election, each table's results were independent of one another. Was that true today? Why or why not? In a real election, should different states' results be treated like independent probability experiments? Why or why not?

Emily Litella was especially upset about the presidential election, for rather unique reasons. Never mind.

Review Your Stuff

11. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 15 minutes** on this. Make sure your question is something that ∗**everyone**∗ at your table can do, and that you expect ∗**everyone**∗ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff

12. *Use a copy of the spreadsheet* to determine the expected number of votes for Blue in today's election. What's the expected number for Red? Wait, *what*? How is this possible?

Looks like Red might demand another recount.

13. You've heard of Pascal's Triangle but probably not Pascal's Tetrahedron! In Pascal's Tetrahedron there is a 1 at the top then *three* numbers under it in a triangular pattern, then *six* under that, then ten. Each number in Pascal's Tetrahedron is the sum of three numbers above it, in fabulous 3-D!

Pascal's Tetrahedron can make you happy when skies are gray . . .

a. Try writing out some of the "levels" (like the triangle's rows) of Pascal's Tetrahedron. Like the triangle, the 1 at the very top is considered "Level 0".

Row 3 has the number 6 in it somewhere, and Row 4 has some consummate 12s.

b. Look for some patterns in Pascal's Tetrahedron, notably patterns within levels. You may need to dig to Level 4 or Level 5 to really see the patterns, but there are some big ones.

. . . but it can't stop a cop on a bike from busting nighttime parties on soccer fields.

c. The numbers in Row n of Pascal's Triangle can be generated by expanding $(h+t)^n$. What expansion could you use to find the numbers in Level n of Pascal's Tetrahedron?

d. How does the expansion connect to some of the patterns you found?

14. Think of your 120 coin flips as 30 four-flip sequences.

a. If your 120 flips are real, how many of the 30 four-flip sequences do you expect to have 0, 1, 2, 3, 4 heads? Fill in this information in the second column of the table below.

Remember that one time at math camp where you had to fake coin flips? Some of you were pretty darn good at that.

b. Find some other sets of coin flips, real or fake. Tally up the number of four-flip sequences that have 0, 1, 2, 3, 4 heads and fill in this information in the latter columns below.

Latter? I hardly know her!

c. Figure out a way to measure how "far" your tallies in each column are away from the theoretical distribution.

	Expected #	Flip Set A	Flip Set B	Flip Set C	Flip Set D
0 heads					
1 head					
2 heads					
3 heads					
4 heads					

15. Use Diana's argument to show that the probability that a randomly chosen integer is *square free* is exactly . . . uhh . . . eh, you figure it out.

Square free means the number has no squares larger than 1 as factors: no 4, no 9, etc. Pascal's Tetrahedron is also square free, because it's made of triangles.

16. David plays Deborah in a best-of-7 series, ending whenever one player wins 4 games. Deborah wins each game with probability $p = 0.6$. What is the expected number of games played in the 7-game series?

17. Deborah wants to play a long enough series with David to give herself a 95% chance of winning the series. How long a series must she and David play to ensure this?

Tough Stuff

18. David plays Deborah in tennis. David wins each point with probability $p = 0.6$. A tennis game is won on the 4th point, but the player must win by 2: 4-3 is not a victory, but 5-3 is.

Ignore the weird scoring in tennis. If David is ahead "30-0" it really means 2 points to 0.

a. If the score is currently 3-3, what is the probability that David wins the game? A tree diagram or conditional probability can be helpful here if you think about it carefully.

b. Determine the probability that David wins the game starting from 0-0.

Remember that one time at math camp where Bill McCallum played Cards Against Humanity? That was awesome.

19. Manipulate

$$\prod_{p \text{ prime}} \left(\frac{1}{1 - \frac{1}{p^2}} \right) \qquad \text{into} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

to show that the infinite product from problem 9 converges to $6/\pi^2$. Hint: Think geometric series.

20. Compute the value of

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) = 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1}$$

I swear I've seen this somewhere.

Group	Coins Used	Votes Up For Grabs	Prob Blue Wins if Red Rolls 3
PC Table 1	5	19	$\frac{3}{16} = 0.1875$
PC Table 2	11	19	$\frac{227}{256} \approx 0.8867$
PC Table 3	12	25	$\frac{3797}{4096} \approx 0.9270$
PC Table 4	10	37	$\frac{53}{64} \approx 0.8281$
PC Table 5	9	25	$\frac{191}{256} \approx 0.7461$
PC Table 6	7	25	$\frac{1}{2} = 0.5000$
PC Table 7	8	31	$\frac{163}{256} \approx 0.6367$
PC Table 8	4	19	$\frac{1}{16} = 0.0625$
PC Table 9	3	66	0
PC Table 10	2	38	0
PC Table 11	14	33	$\frac{7957}{8192} \approx 0.9713$
PC Table 12	13	17	$\frac{3907}{4096} \approx 0.9539$
Las Cruces eTable	6	5	$\frac{11}{32} \approx 0.3438$

Another huge polynomial:

$$\begin{aligned}
& \left(\frac{11b^{5}}{32}+\frac{21r^{5}}{32}\right)\left(\frac{3b^{19}}{16}+\frac{13r^{19}}{16}\right)\left(\frac{227b^{19}}{256}+\frac{29r^{19}}{256}\right)\left(\frac{3797b^{25}}{4096}+\frac{299r^{25}}{4096}\right)\left(\frac{53b^{37}}{64}+\frac{11r^{37}}{64}\right)\left(\frac{191b^{25}}{256}+\frac{65r^{25}}{256}\right)\left(\frac{b^{25}}{2}+\frac{r^{25}}{2}\right) \\
\times\; & \left(\frac{163b^{31}}{256}+\frac{93r^{31}}{256}\right)\left(\frac{b^{19}}{16}+\frac{15r^{19}}{16}\right)\left(0b^{66}+1r^{66}\right)\left(0b^{38}+1r^{38}\right)\left(\frac{7957b^{33}}{8192}+\frac{235r^{33}}{8192}\right)\left(\frac{3907b^{17}}{4096}+\frac{189r^{17}}{4096}\right) \\
=\; & 0.00000004337b^{0}r^{359}+0.00000002271b^{5}r^{354}+0.0000008965b^{17}r^{342}+0.0000003524b^{19}r^{340}+0.0000004696b^{22}r^{337} \\
+\; & 0.0000001845b^{24}r^{335}+0.0000007216b^{25}r^{334}+0.0000003779b^{30}r^{329}+0.00000007601b^{31}r^{328}+0.000001468b^{33}r^{326} \\
+\; & 0.000007324b^{36}r^{323}+0.0000002089b^{37}r^{322}+0.0000008708b^{38}r^{321}+0.000003815b^{41}r^{318}+0.00001502b^{42}r^{317} \\
+\; & 0.00000005324b^{43}r^{316}+0.000005863b^{44}r^{315}+0.000007813b^{47}r^{312}+0.000001571b^{48}r^{311}+0.000003071b^{49}r^{310} \\
+\; & 0.00003327b^{50}r^{309}+0.00001193b^{52}r^{307}+0.0000008231b^{53}r^{306}+0.000004319b^{54}r^{305}+0.00001952b^{55}r^{304} \\
+\; & 0.000002962b^{56}r^{303}+0.000006255b^{57}r^{302}+0.00002443b^{58}r^{301}+0.000002262b^{59}r^{300}+0.000001100b^{60}r^{299} \\
+\; & 0.0001227b^{61}r^{298}+0.000003479b^{62}r^{297}+0.00001448b^{63}r^{296}+0.000002573b^{64}r^{295}+0.00006348b^{66}r^{293}+0.00006206b^{67}r^{292} \\
+\; & 0.000001252b^{68}r^{291}+0.0002668b^{69}r^{290}+0.000007075b^{70}r^{289}+0.000003441b^{71}r^{288}+0.00003155b^{72}r^{287}+0.00006143b^{73}r^{286} \\
+\; & 0.0001391b^{74}r^{285}+0.0005211b^{75}r^{284}+0.000001802b^{76}r^{283}+0.0001985b^{77}r^{282}+0.00003208b^{78}r^{281}+0.00007192b^{79}r^{280} \\
+\; & 0.0003060b^{80}r^{279}+0.00008548b^{81}r^{278}+0.0001040b^{82}r^{277}+0.00009867b^{83}r^{276}+0.00003764b^{84}r^{275}+0.00002588b^{85}r^{274} \\
+\; & 0.0004342b^{86}r^{273}+0.0001603b^{87}r^{272}+0.0001282b^{88}r^{271}+0.0001003b^{89}r^{270}+0.000004142b^{90}r^{269}+0.0002039b^{91}r^{268} \\
+\; & 0.0003399b^{92}r^{267}+0.00004618b^{93}r^{266}+0.004172b^{94}r^{265}+0.0001178b^{95}r^{264}+0.00005726b^{96}r^{263}+0.0001340b^{97}r^{262} \\
+\; & 0.0006703b^{98}r^{261}+0.002159b^{99}r^{260}+0.002142b^{100}r^{259}+0.00004239b^{101}r^{258}+0.0006378b^{102}r^{257}+0.0003494b^{103}r^{256} \\
+\; & 0.0002912b^{104}r^{255}+0.001201b^{105}r^{254}+0.002173b^{106}r^{253}+0.0003380b^{107}r^{252}+0.0004193b^{108}r^{251}+0.0001520b^{109}r^{250} \\
+\; & 0.0001843b^{110}r^{249}+0.001468b^{111}r^{248}+0.002492b^{112}r^{247}+0.001407b^{113}r^{246}+0.001092b^{114}r^{245}+0.00006892b^{115}r^{244} \\
+\; & 0.0001747b^{116}r^{243}+0.002149b^{117}r^{242}+0.0008978b^{118}r^{241}+0.01376b^{119}r^{240}+0.0004769b^{120}r^{239}+0.0001825b^{121}r^{238} \\
+\; & 0.0004423b^{122}r^{237}+0.002079b^{123}r^{236}+0.006917b^{124}r^{235}+0.008966b^{125}r^{234}+0.0003019b^{126}r^{233}+0.0005464b^{127}r^{232} \\
+\; & 0.001013b^{128}r^{231}+0.001187b^{129}r^{230}+0.004647b^{130}r^{229}+0.02277b^{131}r^{228}+0.0003472b^{132}r^{227}+0.001383b^{133}r^{226} \\
+\; & 0.0006205b^{134}r^{225}+0.0004436b^{135}r^{224}+0.01207b^{136}r^{223}+0.01001b^{137}r^{222}+0.004498b^{138}r^{221}+0.003169b^{139}r^{220} \\
+\; & 0.0002193b^{140}r^{219}+0.0001066b^{141}r^{218}+0.006244b^{142}r^{217}+0.006256b^{143}r^{216}+0.01296b^{144}r^{215}+0.001945b^{145}r^{214} \\
+\; & 0.0001335b^{146}r^{213}+0.0005308b^{147}r^{212}+0.003846b^{148}r^{211}+0.005925b^{149}r^{210}+0.02963b^{150}r^{209}+0.0007266b^{151}r^{208} \\
+\; & 0.0003336b^{152}r^{207}+0.0008407b^{153}r^{206}+0.003259b^{154}r^{205}+0.01499b^{155}r^{204}+0.06590b^{156}r^{203}+0.0003683b^{157}r^{202} \\
+\; & 0.001659b^{158}r^{201}+0.001706b^{159}r^{200}+0.0002877b^{160}r^{199}+0.03450b^{161}r^{198}+0.04032b^{162}r^{197}+0.003631b^{163}r^{196} \\
+\; & 0.002628b^{164}r^{195}+0.0001545b^{165}r^{194}+0.00008641b^{166}r^{193}+0.02146b^{167}r^{192}+0.01502b^{168}r^{191}+0.008274b^{169}r^{190} \\
+\; & 0.005339b^{170}r^{189}+0.00001641b^{171}r^{188}+0.0002058b^{172}r^{187}+0.008050b^{173}r^{186}+0.003612b^{174}r^{185}+0.03711b^{175}r^{184} \\
+\; & 0.0004713b^{176}r^{183}+0.0002702b^{177}r^{182}+0.0004921b^{178}r^{181}+0.002295b^{179}r^{180}+0.01798b^{180}r^{179}+0.05461b^{181}r^{178} \\
+\; & 0.0002779b^{182}r^{177}+0.0006436b^{183}r^{176}+0.001202b^{184}r^{175}+0.000003698b^{185}r^{174}+0.02850b^{186}r^{173}+0.1103b^{187}r^{172} \\
+\; & 0.0006764b^{188}r^{171}+0.001539b^{189}r^{170}+0.00001016b^{191}r^{168}+0.05782b^{192}r^{167}+0.009744b^{193}r^{166}+0.006392b^{194}r^{165} \\
+\; & 0.003760b^{195}r^{164}+0.00001156b^{196}r^{163}+0.00002529b^{197}r^{162}+0.005673b^{198}r^{161}+0.002925b^{199}r^{160}+0.01527b^{200}r^{159} \\
+\; & 0.000006058b^{201}r^{158}+0.00003179b^{202}r^{157}+0.0003468b^{203}r^{156}+0.006970b^{205}r^{154}+0.03181b^{206}r^{153}+0.00001665b^{207}r^{152} \\
+\; & 0.00007908b^{208}r^{151}+0.01666b^{211}r^{148}+0.07772b^{212}r^{147}+0.0002805b^{213}r^{146}+0.001084b^{214}r^{145}+0.04074b^{217}r^{142} \\
+\; & 0.0001252b^{218}r^{141}+0.001225b^{219}r^{140}+0.00001782b^{222}r^{137}+0.0003442b^{224}r^{135}+0.001634b^{225}r^{134}+0.0008563b^{230}r^{129} \\
+\; & 0.02241b^{231}r^{128}+0.00005572b^{233}r^{126}+0.01174b^{236}r^{123}+0.00002919b^{238}r^{121}+0.001152b^{250}r^{109}+0.0006034b^{255}r^{104}
\end{aligned}$$

Oh man, I’m glad that’s over.