Problem Set 14: Roll Out

Opener

The contestant wins today's game by rolling five dice. Any 4, 5, 6 is considered a "win" but all five dice must win. The player gets to re-roll dice twice, like in Yahtzee.

And so we've come . . . to the *end* of the *roooooooad* . . .

1. After the first roll, what situations can the contestant be in, and with what probabilities?
2. After the second roll, what situations can the contestant be in, and with what probabilities?

Still I can't stop rolling. It's unnatural.

3. Compute the probability that the contestant wins the game.
4. One of the five dice was blue. What is the probability that in three rolls, the blue die will land on 4, 5, or 6 at least once?
5. Use problem 4 to compute the probability that the contestant wins the game.

Important Stuff

6. Make up a situation that each of these polynomials could model.
   a. $(a+b+c)^6$
   b. $\left(\frac{x}{6}+\frac{x^2}{6}+\frac{x^3}{6}+\frac{x^4}{6}+\frac{x^5}{6}+\frac{x^6}{6}\right)\left(\frac{x^3}{2}+\frac{x^{-3}}{2}\right)^2$
   c. $\left(\frac{x}{3}+\frac{2x^{-1}}{3}\right)^2\left(\frac{x}{4}+\frac{3x^{-1}}{4}\right)^2$
   d. $\left(\frac{x^2}{12}+\frac{5}{12}+\frac{6x^{-2}}{12}\right)^2$

These polynomials are made for modelin', and that's just what they'll do. One of these days these polynomials are gonna model Brian Shay.

7. Over the last few weeks, sometimes you added probabilities, sometimes you multiplied them. When did you and why can you add and multiply probabilities?

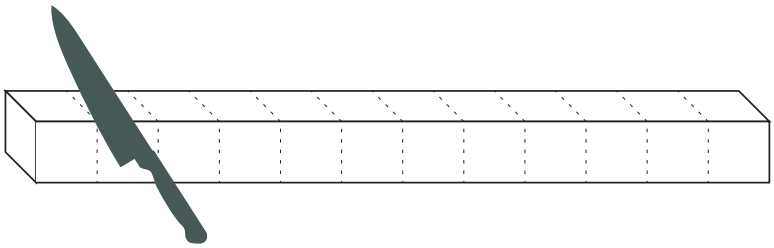
Barb and Monica say that if you don't know this by now, you're never gonna get it never gonna get it, never gonna.

Unimportant Stuff

8. We hid the eight mathematical practices in previous side notes and problems. Recite the eight practices from memory, then find the references.

When it comes to practices, Bill McCallum ain't got nothin' to do with my selection. Prove stuff using appropriate tools? Only if it's practices 5 and 3.

9. Nate has a block of cheese that is 12 inches long. Moving from left to right 1 inch at a time, he flips a fair coin to decide whether to... er... cut the cheese.



a. Create a histogram for the probability of Nate getting a certain number of pieces of cheese.
b. What is the mostly likely number of pieces of cheese that Nate will cut?

Darryl, are you sure this isn't a histograph?

Your Stuff

Disclaimer: Bowen and Darryl would like to remind everyone that they didn't write the problems or sidenotes in this section, especially the sidenote about Dave.

[Except for these. These notes are in our HANDS, in our HANNNDS . . .]

NM. Troy and Gail are playing a game with dice. Troy has a six-sided die (d6) and Gail has a number of three-sided dice (d3). Troy rolls his die to set a target number for Gail. Gail wins if she rolls at least as many ones as the number on Troy's die. Troy rolls a two.

a. What is the probablity that Gail wins if she has five dice?
b. What is the probablity that she wins if she has six dice?
c. What is the probability that she wins if she has six dice and Troy rolls a three?

Dice Dice Baby Las Cruces Dice Dice Baby
Now the polynomials are jumping
With the exponents kicked in, the techmology is pumpin'
Quick to the solution, to the solution no faking...

1. Three days before ovulation, the Lego My Prego test gives the following results. If a woman is pregnant, the test is positive 51 percent of the time. If a woman is

[Really, Table 1? Really??]

not pregnant, the test is negative 88 percent of the time. The Lego My Prego test is more accurate seven days after a woman's missed period. If a woman is pregnant, the test is positive 96 percent of the time. If a woman is not pregnant, the test is negative 99 percent of the time. Daisy Duke met up with Boss Hogg at High West and both had a few too many whiskeys. She takes a pregnancy test 3 days before ovulation and 7 days after her missed period. What is the chance that Hogglet is on the way given that . . .

[It's just like Gabe warned us, the Devil went down to Georgia . . .]

a. . . . both tests come back positive?

b. . . . the first test comes back negative and the second test comes back positive?

2. **Pascal's Dartboard** Annemarie and Caitlin go to Cisero's to play darts. After a renovation, Cisero's now has a new dartboard. It is a triangular dartboard with 10 rows of Pascal's Triangle. Two points are given for every even number hit, 9 points are given for every prime number hit. If you hit a 2 you get 18 points! Caitlin starts and throws three darts.

"Hey Annemarie! You're married to Dave . . . the CrossFit guy? I hear he was a terrible table leader!"

[You have to assume Caitlin hits the dartboard all 3 times, which, I guess. Caitlin isn't interested in darts anymore, she just wants to dance with somebody.]

   a. What are all the possible point outcomes of Caitlin's three dart throws?
   b. What is the probability Caitlin will get an even number? Prime number? Three even numbers in a row? Three primes in a row?
   c. What is the expected value of points for Caitlin after her first 3 darts?
   d. (Tough) If the dartboard was infinite, what is the expected value of points for Caitlin after her first 3 darts?

3. 60% of the singers at Karaoke Night were PCMI participants and 40% were locals. Of the PCMI participants, 90% tipped Cowboy Cal. Of locals, 50% tipped Cowboy Cal.

Cowboy Cal! He's branching out.

   a. Of those who tipped Cowboy Cal, what proportion were PCMI participants?
   b. Of PCMI participants, what proportion tipped Cowboy Cal?

4. Write the first letter of the first names of each participant at your table, then determine the number of rearrangements of the letters.

5. What is the probability that a number in a row of Pascal's Triangle is odd? You'll need to find a pattern to the number of odd numbers in each row first!

[Carl was thinking about this problem while falling in love at the Copa.]

5. There are two playing cards, a joker with a normal "back" side and a mistake card with two "back" sides printed. The cards are placed in a bag. You draw one and see a "back" side. What is the probability that the other side of the same card is also a "back" side?

[The probability of getting back? The answer is clearly "much".]

6. It was Cowboy Karaoke night at Cisero's last night. Of the people chosen to sing, 90% tipped Cowboy Joe. Of the songs performed, 25% were chosen by the singer and 75% were chosen by others. 20% of the singers were booed. Assume all three events were independent.

[This problem required us to boo at 20% of the songs, especially "Hang On Sloopy".]

- **a.** Of the eight possible outcomes (2^3 for tipping, choosing, and booing), what was the most frequent outcome and how frequent was it?
- **b.** Write a polynomial multiplication that models the situation (you don't have to multiply it out).
- **c.** Of the singers that were booed, what proportion picked their own songs?

7. In a single elimination four team tournament, the winners advance. In the first round, the 1-seed faces the 4-seed and the 2-seed faces the 3-seed. Given that the probability of a team winning is the opponent's seed over the sum of the two seeds . . .

- **a.** What is the probability of the 1-seed winning the tournament? What is the probability of the 4-seed winning?
- **b.** What if it was double elimination? (Winners bracket and losers bracket)
- **c.** Which format is more beneficial for the 1-seed? 2-seed? 3-seed? 4-seed?

8. There are three breakfast options here at PCMI: cold breakfast, eggs with bacon, eggs with ground meat. The

probability of cold breakfast and the probability of eggs with bacon are the same and both of these are twice as likely as bacon with ground meat. Fred only likes hot breakfasts. If Fred goes to breakfast three times this week, how likely is it that he will get a hot breakfast all three times? Two times?

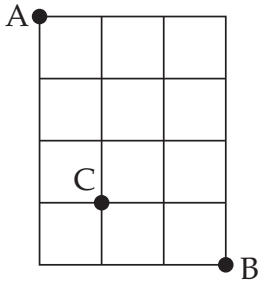
Maybe they should serve fdizzle for breakfast!

If Fred loses he can cry if he wants to.

9. A woman in her 40s has roughly a 1% chance of having breast cancer. A mammogram tests for breast cancer. If a woman has breast cancer, the test is positive 90% of the time and negative 10% of the time. If a woman does not have breast cancer, the test is positive 10% of the time and negative 90% of the time.

- **a.** (Old) What is the probability that a woman has breast cancer, given that the mammogram came back positive?
- **b.** (Old, but in Neat Stuff) What is the probability that a woman has breast cancer, given that two mammograms have come back positive?
- **c.** (New) How many positive mammograms would be needed to have a 90% probability of having breast cancer?

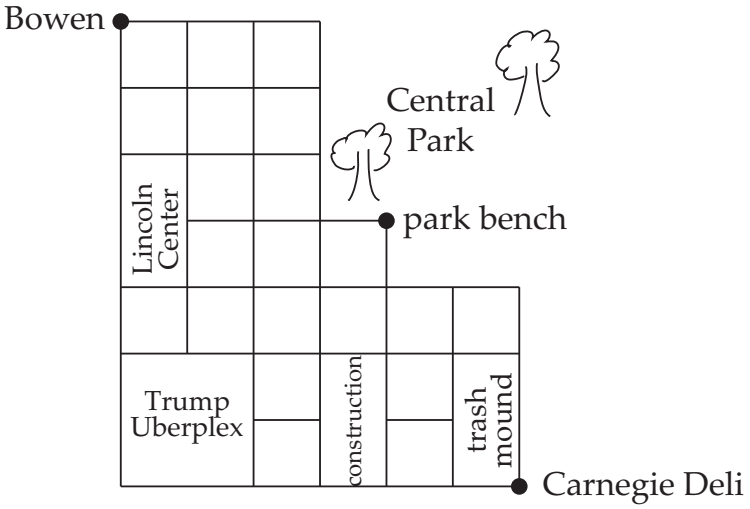
10. Starting at point A, you are trying to get to point B by following a path. You can only go down and to the right (so you can't go backwards or over the same path).



- **a.** How many possible ways are there to get from A to B?
- **b.** Assuming you choose a path at random, what's the probability you go past your friend's house at C?

[Girl, look at them numbers. They work out!]

10. Bowen wants to get his sandwich on at Carnegie Deli.



a. How many ways can he take to get there, assuming he will only walk south or east?

b. Bowen heard there is a little 3-card monte happening at the park bench in Central Park.

(i) Assuming he picks paths randomly, what's the probability he gets to the 3-card monte game on his way to the Deli?

(ii) What's the probability he *wins* 3-card monty?

[Bowen's hoping to earn some bills, bills, bills, but is more likely to become a man of constant sorrow.]

11. In Yahtzee, you get three rolls and you're looking to get all 5 dice to be the same number (called a "Yahtzee"). You can "save" dice from one roll to the next.

a. Find the probability that you roll a Yahtzee on your first roll.

b. Find the probability that you roll a Yahtzee on your second roll. (You will probably want some "cases".)

c. (Harder!) Find the probability that you roll a Yahtzee on your third roll. (You will need to carefully consider the case that you roll a pair on Roll 1 then a different three of a kind on Roll 2.)

11. In the opening game on Set 7, Wendy placed four tags next to four items. There was a 9/24 chance for her to walk away empty-handed.

a. If there were 2 items instead of 4, what's the probability that Wendy wins nothing?

Fortunately, Wendy won 1 item, or she might have become deranged.

b. What if there were 3 items?
c. What if there were 5 items?
d. Predict the answer for 6 items based on any patterns in the previous data.
e. Which is more likely: winning 1 item or winning no items? Check for each case.

12. Darryl deals Bowen a 5-card poker hand. After Darryl has dealt two cards, Bowen sees that he has aces. What is the probability that the next three cards will give Bowen . . .

a. . . . exactly 3 aces?

b. . . . 3 or more aces? (Assume no jokers.)

12. Molley is adventuring with her trusty puppy. A goblin jumps out and grabs her puppy. Molley pulls out her sword (the magic sword of solving by parts) and salleys forth to slay the goblin. If she rolls a sum of 17 or 18 with three six-sided dice (d6) she gets to roll two four-sided dice (d4) for damage. If she rolls a 9 or higher sum on three d6 she rolls one d4. If she rolls less than 9 she misses. If she totals 7 or higher damage points the goblin is knocked out. How likely can she get him with one swing?

[Molley sees the goblin with the dog she likes, and she's like, forget you.]

Tough Stuff

5. The Fibonacci sequence goes 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... Each term is the sum of the two before it.
a. Find the probability that a Fibonacci number is a multiple of 2.
b. Find the probability that a Fibonacci number is a multiple of 3 . . . of 5.
c. Find the probability that two randomly chosen Fibonacci numbers share a common factor of 2.
d. Find the probability that two randomly chosen Fibonacci numbers do *not* share a common factor of 2.
e. Under what circumstances is a Fibonacci number divisible by a specific prime?

f. Find the probability that two randomly chosen Fibonacci numbers do not have any common factors greater than 1.

14. Imagine that we had coin flip data sets that were much longer than 120 flips. Here's a possible test to determine if a set might be real or fake: Partition the data set into groups of $2m$ flips. For each group, let a be the binary number represented by the first m coin flips plus one, and b be the binary number represented by the latter m coin flips plus one. Notice $1 \leqslant a, b \leqslant 2^m$. Calculate the proportion of the $2m$-flip groups that have $\gcd(a, b) = 1$. If the data set is truly random, this proportion should approach $6/\pi^2$ as the length of the data set and m go to infinity.

a. What is the theoretical distribution for this proportion when m and the data set length are finite?

b. So, 120 isn't a very long set of coin flips. Investigate the feasibility of this test for various m.

15. Consider a right rectangular prism whose height is h cm and whose square base has area 1 cm^2. If $h = 1$, the prism is a perfect cube and the probability that any one of the faces appears at the top when the dice is tossed is 1/6. What are the probabilities of each face of the prism to appear if h is close to, but not exactly, 1? What if the base of the prism is also not a square?

You give dice . . . a bad name.

No More Stuff

16. Thanks. See you again soon.