Rethinking Fractions: Implications for Teaching and Learning Algebra

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Research on learning algebra: 
*Making links to the classroom*

**1988 NCTM Yearbook on Algebra: Common Mistakes in Algebra** (Marquis, 1988)

\[
a^2 \cdot b^5 = (ab)^7
\]
\[
\frac{x+y}{x+z} = \frac{y}{z}
\]
\[
(x+4)^2 = x^2 + 16
\]
\[
\frac{x}{y} + \frac{r}{s} = \frac{x+r}{y+s}
\]
\[
3a^{-1} = \frac{1}{3a}
\]

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What is the meaning of this fraction? 

What is the meaning of the top number? 

What is the meaning of the bottom number? 

Draw a visual model to represent it.
A fraction

Is typically thought of as:

- Quotient,
- Part to Whole, or
- Ratio

Rethinking Fractions:

Based on Part 2, Fractions by H. Wu
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A fraction $\frac{1}{b}$ is the quantity formed by 1 part when a whole is partitioned into $b$ equal parts; a fraction $\frac{a}{b}$ is the quantity formed by $a$ parts of size $\frac{1}{b}$.

- A fraction is a number on the number line.

2. a. Represent a fraction $\frac{1}{b}$ on a number line by defining the interval from 0 to 1 as the whole and partitioning it into $b$ equal parts. Recognize that each part has size $\frac{1}{b}$ and that the endpoint of the part based at 0 locates the number $\frac{1}{b}$ on the number line.

b. Represent a fraction $\frac{a}{b}$ on a number line by marking off $a$ lengths of $\frac{1}{b}$ from 0. Recognize that the resulting interval has size $\frac{a}{b}$ and that its endpoint locates the number $\frac{a}{b}$ on the number line.

CCSS, 2010
Grade 3:

3. equivalence of fractions in special cases, and compare fractions by reasoning about their size.

a. two fractions are equivalent (equal) if they are the same size, or the same point on a number line.

b. Recognize and generate simple equivalent fractions. Explain why the fractions are equivalent,

c. Express whole numbers as fractions, and recognize fractions that are equivalent to whole numbers.

d. Compare two fractions with the same numerator or denominator by reasoning about their size. Recognize that comparisons are valid only when the two fractions refer to the same whole.

CCSS, 2010
Grade 4: Equivalence, comparing

Extend understanding of fraction equivalence & ordering.

1. Explain why a fraction $a/b$ is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models. Use this to recognize and generate equivalent fractions.

2. Compare two fractions with different numerators and denominators, e.g., by creating common denominators or numerators, or by comparing to a benchmark fraction such as 1/2. Recognize that comparisons are valid only when the two fractions refer to the same whole.

Build fractions from unit fractions by applying /extending previous understandings of whole number operations.

CCSS, 2010
People learn by

- Engaging in a concrete experience
- Observing reflectively
- Developing an abstract conceptualization based upon the reflection
- Actively experimenting/testing based upon the abstraction

Zull, 2002
Conceptual Knowledge:
- Makes connections visible,
- enables reasoning about the mathematics,
- less susceptible to common errors,
- less prone to forgetting.

Procedural Knowledge:
- strengthens and develops understanding
- allows students to concentrate on relationships rather than just on working out results

NRC, 1999; 2001
Conceptual understanding

- Take an action on a mathematical object
- Observe the mathematical consequences and
- Reflect on the mathematical implication of those consequences
Interactive Dynamic Technology
Action Consequence Documents
The only reasons to ask questions is to: (Black et al., 2004)

Probe to uncover students’ thinking
discover misconceptions that exist

Push to advance students’ thinking
make connections
justify or prove their thinking.
Unit fraction & the number line

Fraction activity 1, 2
Equivalent fractions

Fraction activity 3
Grade 4: Equivalence, equality, order

Fraction Activity 3
Generating equivalent fractions

\[ \frac{1}{2} = \frac{3 \times 1}{3 \times 2} \]

D = 2
What is a fraction?

Describe where three fifths will be. How will three fifths differ from seven fifths? Explain your thinking, then check your answer using the tns file.

Where will 4/8 be? b) 0/8? c) Is eleven eighths closer to one or to two? How do you know?

If the number of 1/5’s is larger than the 5, what can you say about the size of the fraction? Explain.

Suppose the unit fraction was 1/5 and the numerator was between 11 and 14. Where is the fraction?

If the unit fraction were 1/6, where would fractions with a numerator between 25 and 29 be?
What is a fraction?

- How many copies of \( \frac{1}{2} \) are in 2?
- Use the file to make a conjecture about whether the following sentences are correct.
  
  a) 0 is a fraction.
  
  b) A whole number cannot be a fraction.
  
  c) A fraction can have many names.
What does fraction as a point on a number line buy us?

- A constant way to think: $k/p$ is $k$ copies of $1/p$ - the length of the concatenation of $k$ segments each of which has length $1/p$.
- Behavior similar to whole numbers:
  - $k/3$ is a multiple of $1/3$
  - Larger fraction is to the right on the number line.
- Connection of whole number to fractions.
- One number has many names and none more important than another.
- No difference between proper and improper fractions.
Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.
Unit squares

- The area of the unit square is by definition 1.
- If two regions are congruent, then their areas are equal. (If two shapes are congruent, then one figure can be rotated, translated or reflecting to fit on exactly on top of the other.)

- Find four representations of 1/4.
- Find two representations of 5/6, 7/4, 9/4.
Grade 3: Fraction as area

Fraction activity 4
Grade 4: Multiplication

3. A fraction a/b with a > 1 is a sum of fractions 1/b.
   - Addition and subtraction of fractions is joining and separating parts referring to the same whole.
   - Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation.
   - Add and subtract mixed numbers, like denominators,

4. Apply and extend previous understandings of multiplication to multiply a fraction by a whole number.
   - A fraction a/b is a multiple of 1/b. 5/4 = 5 \times (1/4).
   - A multiple of a/b is a multiple of 1/b, and use this understanding to multiply a fraction by a whole number.

CCSS, 2010
Grade 5: Addition/Subtraction

Use equivalent fractions as a strategy to add and subtract fractions.

1. Add and subtract fractions with unlike denominators by using equivalent fractions to produce an equivalent sum or difference of fractions with like denominators.

2. Solve word problems involving + and - of fractions by using visual fraction models or equations to represent the problem.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

3. Interpret a fraction as division of the numerator by the denominator \( \frac{a}{b} = a \div b \).

4. Apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction.
Grade 5: Multiplication & Area

a. Interpret the product \((a/b) \times q\) as a parts of a partition of \(q\) into \(b\) equal parts; equivalently, as the result of a sequence of operations \(a \times q \div b\). In general, \((a/b) \times (c/d) = ac/bd\).

b. Find the area of a rectangle with fractional side lengths by tiling it with unit squares of the appropriate unit fraction side lengths, and show that the area is the same as would be found by multiplying the side lengths. Multiply fractional side lengths to find areas of rectangles, and represent fraction products as rectangular areas. 

CCSS, 2010
Grade 6: Division

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.

CCSS, 2010
Adding Fractions
Whole number $x$ a fraction

Fraction activity 2
Grade 5: Multiplying fractions

Fraction Activity 5
Division of fractions: \( \frac{2}{3} \div \frac{3}{4} \)
A unit other than 1

If the weight of an object X is our unit, then 5/7 would mean 5 parts of X after it has been partitioned into 7 parts of equal weight.

Unit is the weight of a piece of ham that weighs three pounds. 1/3 of the weight would be one pound.

Unit is a pack of 15 pencils; 2/3 of a unit would be to partition into three equal parts of 5 pencils or 10 pencils.

Wu, 2011
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