Permutation groups and representation theory

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The group S_3

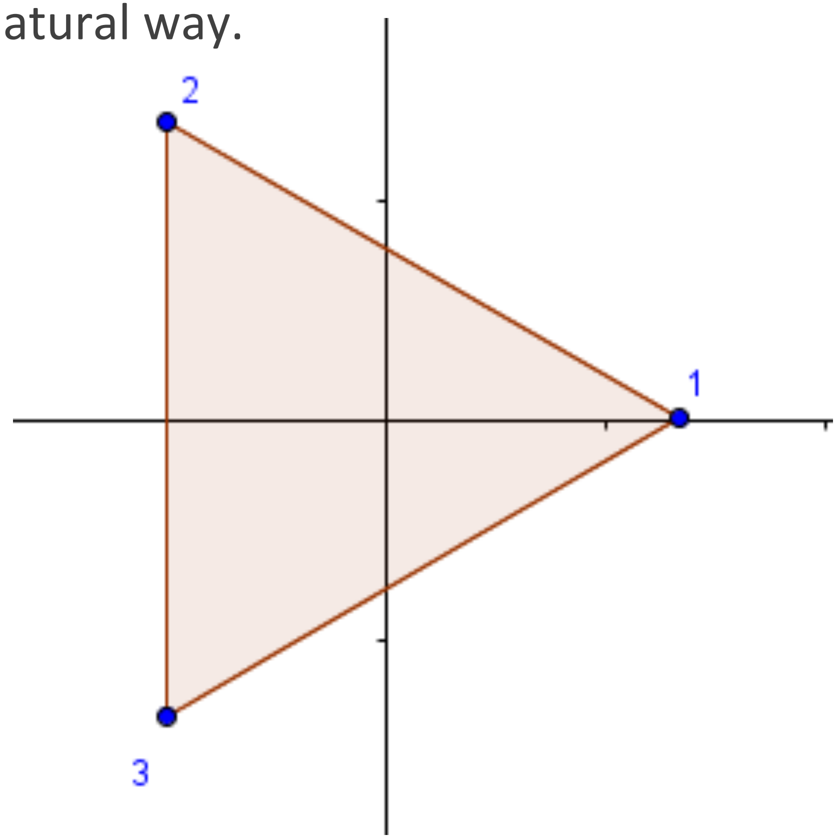
Group: A set of things, and an operation on that set. For example, “(Integers, +)”, i.e. integers under addition, form a group.

In S_3, our elements are *permutations* of three symbols. So, not the three symbols themselves, but the different ways you can permute those symbols.

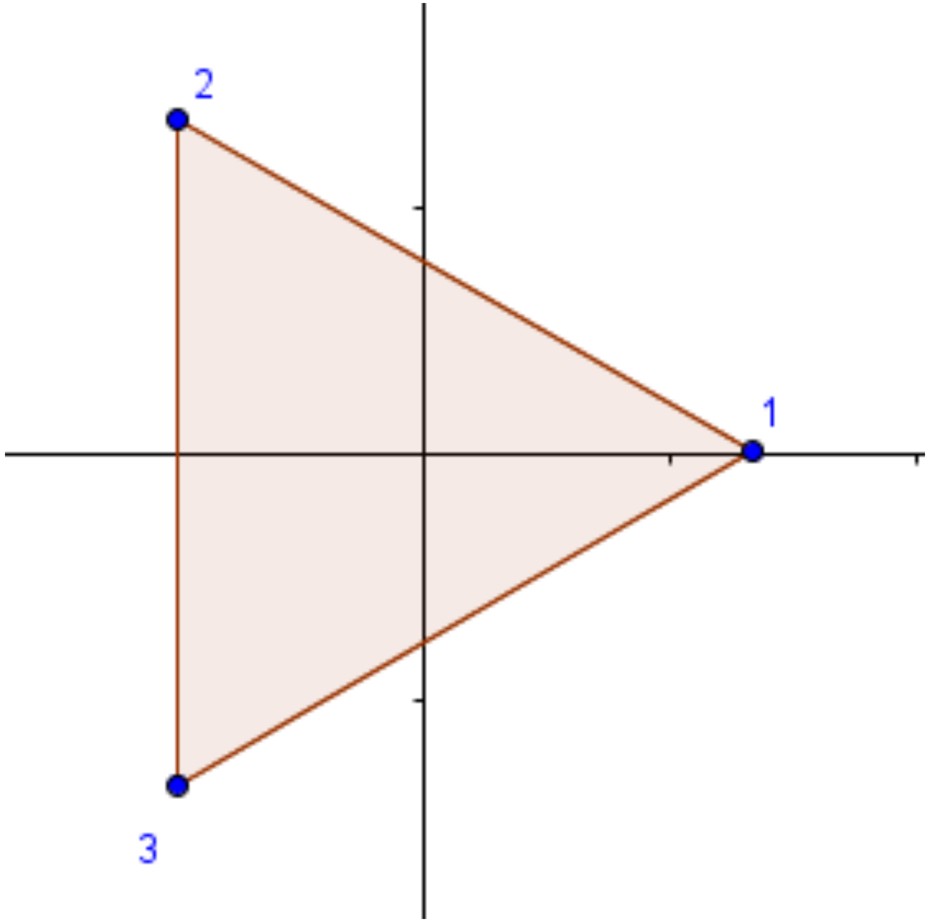
Example: (1 3)(2).

Group actions (on a space)

There's a lot to this that I'm eliding over … but we can think about this group acting on the Cartesian plane (R^2) in a very natural way.



“Symmetries of the equilateral triangle”



S_3 acting on another space

Suppose we have a 3-dimensional space, with basis elements $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

The element (1 2) in S_3 can act on this by sending e_1 to e_2 and e_2 to e_1, and leaving e_3 alone.

$(1\ 2)\ (5e_1 – 3e_2 + e_3) = (-3e_1 + 5e_2 + e_3)$

Representation theory

Can we describe these group actions in another fashion? That fashion should also preserve structure.

Answer:

(This is useful if we're very comfortable with, and/or know a lot about, our alternative description. And that happens here.)

Trivial representation

	e	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
X_1	[1]	[1]	[1]	[1]	[1]	[1]

S_3 acting on another space

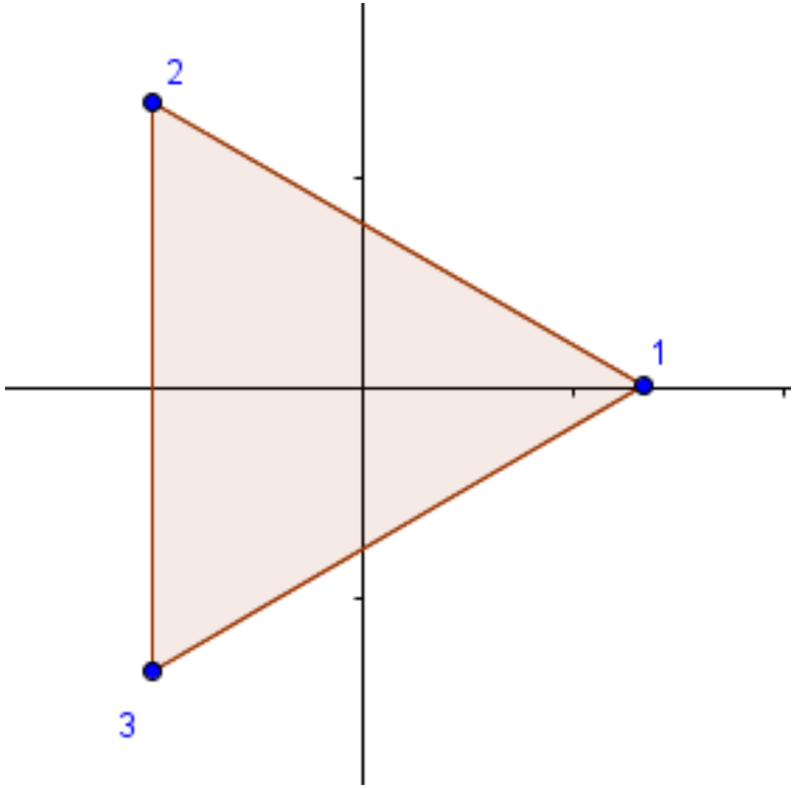
Suppose we have a 3-dimensional space, with basis elements $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

Note that every element in our space can be written as a linear combination of the basis elements.

The element (1 2) in S3 can act on this by sending e_1 to e_2 and e_2 to e_1, and leaving e_3 alone.

Associate with this (1 2) action a matrix that captures the behavior of (1 2).

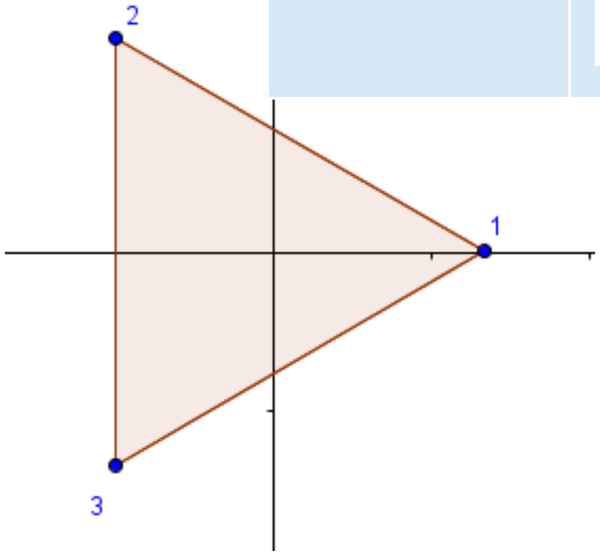
“Symmetries of the equilateral triangle”



“Symmetries of the equilateral triangle”

“Standard representation”

	e	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
X_3	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 1\\ -1 & 0\end{pmatrix}$



Further questions and explorations

All this, for groups other than permutation groups. Finite and infinite.

For a given group, can we figure out how many irreducible representations there are?

Can we find the representations?

What structure do the representations themselves have? (Characters.)

What does the representation structure tell us about the original group structure?

Irreducible representations.

Neat fact

How many irreducible representations of S_n are there?

P(n), i.e. the partition number for n. (Remember box counting?)