# Day 2: Everybody's Working For The Weekend

## Opener

**1.** **a.** Today, each person will flip a coin until all eight of the three-flip sequences appear, keeping track of when each sequence appears for the first time. For example, if you flipped . . .

| H | H | H | H | T | T | H | T | H | ··· |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |

then you would write that you first got HHH on the third flip, HHT on the fifth flip, HTT on the sixth flip, and so on. Keep flipping until you've obtained all eight three-flip sequences. Write when each sequence first occurs in the left table.

Everyone's watching, to see what you will do.

The Great Genevieve predicts that you will be writing a 3! And probably a 4!

We apologize to those who get stuck writing an especially large number in the table.

| | |
|---|---|
| HHH | |
| HHT | |
| HTH | |
| HTT | |
| THH | |
| THT | |
| TTH | |
| TTT | |

| | |
|---|---|
| HHH | |
| HHT | |
| HTH | |
| HTT | |
| THH | |
| THT | |
| TTH | |
| TTT | |

**b.** Repeat the experiment again. Fill in the table on the right.

**c.** Combine your data with your tablemates in some meaningful way. One person should enter the combined data here:

http://bit.ly/pcmi2016

Everybody wants a second chance.

Note: not literally "here". There.

### Important Stuff

**2.** Here are some experiments with coins! Try them!

**a.** Evelyn's ready to flip 5 coins. What is the average number of heads she should expect?

**b.** Gabriella's ready to flip a coin as many times as necessary until seeing heads. What is the average number of flips she should expect?

**c.** Nestor's ready to flip a coin as many times as necessary until seeing both a head and a tail at least once. What is the average number of flips he should expect?

**3.** Iceland soccer fans have a new tradition. Each person in the crowd has three cards: blue, white, or red. Each person picks up a card to start. Then there is a series of claps: on each clap, everyone drops their card and randomly picks up one of the two cards they weren't holding. At the start, 60% of the crowd holds blue cards, 30% hold white cards, and 10% hold red cards.

The 10% holding red cards were immediately ejected from the game. (But they're still in the crowd!)

**a.** What would you predict for the distribution of cards after the first clap?

**b.** . . . after the second clap?

**c.** . . . after the third clap?

**d.** Predict what you think the distribution of cards will be after 10 claps.

Fun fact: 10% of the population of Iceland is currently in France! Seriously, it's true.

**4.** Repeat problem 3, but start with everyone holding blue cards at the start.

**5.** Here's another recursive definition.

$$T(n) = \begin{cases} 600, & \text{if } n = 0 \\ \frac{1}{3} \cdot T(n-1), & \text{if } n > 0 \end{cases}$$

Complete this table for $T(n)$ and the running total of all $T(n)$, keeping your answers accurate to two decimal places.

| n | T(n) | Total |
|---|---|---|
| 0 | 600 | 600 |
| 1 | | 800 |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

Everyone's hopin it'll all work out.

**6.** The sequence $g(0), g(1), g(2), \ldots$ is defined recursively by

$$g(n) = 3g(n-1) + 10g(n-2), \qquad \text{if } n > 1.$$

Six different starting pairs for $g(0)$ and $g(1)$ have been chosen.

| | | | | | | |
|---|---|---|---|---|---|---|
| $g(0)$ | 1 | 0 | 1 | 5 | 0 | 2 |
| $g(1)$ | 0 | 1 | 1 | 0 | 10 | 10 |
| $g(2)$ | 10 | | | | | |
| $g(3)$ | 30 | | | | | |
| $g(4)$ | | | | | | |
| $g(5)$ | | | | | | |

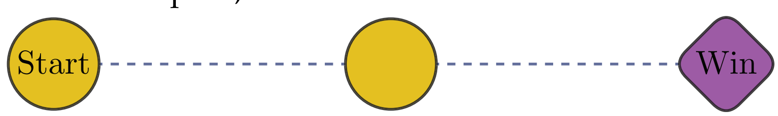
Complete the table above without any technology. Look for patterns to make your work easier. Share patterns with your tablemates.

For example, $g(2)$ is 3 times $g(1)$ plus 10 times $g(0)$. Then . . .

While debatable, writing utensils and paper do not qualify as technology.

## Neat Stuff

**7.** Cesar starts in the circle marked "Start" below. He gets to flip a coin as many times as he wants. When he flips a head, he moves forward one space. When he flips a tail, he moves back one space (if possible). If he reaches the "Win" space, he wins!




What is the probability that Cesar *won't* have won after 8 flips?

You want a piece of my heart? You better start from the Start.

Everyone wants you to come through! And then we'll all say ... Hail Cesar. Sorry.

**8.** Zachary's ready to roll a die as many times as necessary until he gets a six. What is the average number of rolls he should expect?

**9.** **a.** Write a recursive definition for the Fibonacci sequence, which starts with $F(0) = 0$ and $F(1) = 1$.

**b.** Determine $F(0)$ through $F(9)$ and the sum of these ten numbers.

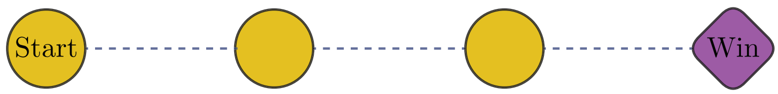
**c.** Pick some different starting pairs for $F(0)$ and $F(1)$, then calculate the same sum. What do you notice?

In the Fibonacci sequence, each term is the sum of the two that come before it. $F(2) = 1$ and $F(3) = 2$.

**10.** The *Lucas sequence* is like the Fibonacci sequence, with the starting pair of 2 and 1. Find as many relationships as you can between Lucas numbers and Fibonacci numbers. Try to prove them!

$L(2) = 3, L(3) = 4, L(4) = 7$. There's a lot of literature on Fibonacci and Lucas; we humbly request that you not read it since you may prefer the chance to find and prove some fun results on your own.

**11.** Mariah plays a game just like Cesar's game, but with more spaces!




What is the probability that Mariah *won't* have won after 8 flips?

If she wins, do we say Ave Mariah? I knew someone who lived on Ave Mariah.

**12.** Emma's about to repeat the experiment from the Opener one more time. What is the probability she will write a 4 in one of the eight spots in the table?

Flip me baby one more time. (Wrong song!)

**13.** Which Fibonacci numbers are even, and which are odd? Explain why this happens.

**14.** Beverly will flip a coin as many times as she needs to until seeing at least 2 heads *and* at least 2 tails. What is the average number of flips she should expect from this experiment?

**15.** TJ flips coins for cash! He gets $1 every time he flips tails. But, if he flips heads twice in a row, he "busts" and loses all his money (but continues playing). The game lasts 10 flips.

You wanna be in the show? Come on, baby, let's go!

**a.** What's the probability that TJ never busts?
**b.** What is the average amount of money TJ expects to win?
**c.** What happens in a longer game? Will the average payout increase or decrease?

### Tough Stuff

**16.** What is the expected number of flips it should take to see all eight of the three-flip sequences?

Everybody's goin off the deep end . . .

**17.** Experiment with the recursion $t(n) = 2x \cdot t(n-1) - t(n-2)$ with the starting pair $t(0) = 1$, $t(1) = x$. Find the zeroes of the $t(k)$ polynomials.