Day 5: Bop, Set, Coset

Opener

1. Set Theory and Awesome-Totes are playing more volleyball. Detailed analysis of their games reveals that:

- when Set Theory serves, they win 90% of the points
- when Awesome-Totes serves, they win 70% of the points

Awesome-Totes gets the first serve. The team that wins a point serves the next one.

Set Theory's volleyball hits: Bop, Set, Jonze.

Awesome-Totes' volleyball hits: Mmmbop, Set 1, Set $1\frac{1}{2}$, Set $1\frac{3}{4}$, Set $1\frac{7}{8}$, . . .

We are using "rally scoring" here. Either team can win a point regardless of who served.

a. Find the probability that Set Theory wins the *second* point.

b. Find the probability that Set Theory wins the *third* point.

c. Multiply this. What does the answer mean in this context?

$$\begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

d. Multiply your answer above by the same matrix:

$$\begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

What does the answer mean in this context?

e. Multiply this. What happens?

$$\begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

f. Find the probability that Set Theory wins the *sixteenth* point.

TI-Nspire hints:

Press up and [enter] to reuse a previous calculation.

Use "Ans" via [ctrl] [(–)] to refer to the previous answer.

Copy, paste, and undo commands work the way you expect them to.

Use [^] if you want to raise something to a power.

Use [carrot] if you want to eat something orange and pointy.

Important Stuff

2. Sometime in the near future, Jacob and Jacob will discover Planet SA, populated by aliens who live on two continents: Continent S and Continent A. Once a year, a land bridge appears by Chance that allows aliens to migrate from one continent to the other. Detailed analysis of their migration will reveal that:

- 10% of Continent S residents move to Continent A each year (others remain on Continent S)
- 30% of Continent A residents move to Continent S each year

Jacob's rise to fame and fortune began years ago, standing in front of a crowd at Pizza and Problem Solving.

Note that all the aliens on Planet SA live an in-continent lifestyle.

At the time of discovery, half the aliens live on Continent S and half on Continent A. Let $S(n)$ and $A(n)$ be the proportions of the population that live on each continent n years after Jacobs first discover the planet.

All this planet needs is another Continent A, and then it would be congruent to Planet XYZ.

a. Write recursive definitions for $S(n)$ and $A(n)$ in terms of $S(n-1)$ and $A(n-1)$.

$$S(n) =$$
$$A(n) =$$

b. Calculate the population proportions 1, 2, 3, and 10 years after the discovery of Planet SA.

c. What must be true about the sum of $S(n)$ and $A(n)$ for all n? Explain why.

The application to visit this planet is really difficult, especially the SA question!

d. After many years, the population proportions appear to approach specific values. What are these "steady state" proportions?

e. At the steady state, what is the relationship between $S(n)$ and $S(n-1)$?

f. Arielle wants to calculate the answers to part d, but without using recursion or matrices. At your tables, come up with a non-recursive strategy using the equations above to do so, then make it happen!

3. In the Opener from Day 4, we defined $A(n)$, $B(n)$, and $C(n)$ as the proportion of olde-tymey PCMI participants at Tables A, B, and C after n table changes. You wrote down these equations:

$$A(n) = 0.50\,A(n-1) + 0.50\,B(n-1) + 0.50\,C(n-1)$$
$$B(n) = 0.16\,A(n-1) + 0.40\,B(n-1) + 0.50\,C(n-1)$$
$$C(n) = 0.34\,A(n-1) + 0.10\,B(n-1)$$

a. Before the first table change, 60% of participants sat at Table A, 10% at Table B, and 30% at Table C. After five table changes, what % of participants are at Table A? Table B? Table C? After 10 table changes?

b. Repeat part a assuming that before the first table change, all of the participants are at Table A before the first table change.

Table A must have been really large! Either that, or people were smaller back then.

c. Use your non-recursive strategy from Problem 2 to calculate the "steady state" proportions.

4. Multiply these. Use the Nspire if you would like to, but look for ways to be lazy.

These numbers might seem familiar . . . Also, this is a good time to let you know you that on the Nspire, you can store a matrix or vector as a variable!

a. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

c. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

d. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

e. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 40 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

g. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

h. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

i. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

j. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

k. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 11 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

l. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix}^2 \begin{bmatrix} 10 \\ 40 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

On National Macaroni Day we celebrate those tiny virtual pets that you had to feed. Or do we celebrate the guy who made the sequence about the rabbits? Wait, maybe it's the guy who says "How you doin" a lot on Friends. No, no, all of this is wrong. It's the day we celebrate feathers and those Drake's cupcake thingies.

5. Multiply these. Again, look for ways to be lazy.

a. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 10 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

c. $\begin{bmatrix} .5 & .5 & .5 \\ .16 & .4 & .5 \\ .34 & .1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

d. $\begin{bmatrix} .5 & .5 & .5 \\ .16 & .4 & .5 \\ .34 & .1 & 0 \end{bmatrix} \begin{bmatrix} .6 & .5 \\ .1 & .286 \\ .3 & .214 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$

6. Calculate these:

a. $1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots =$

b. $\frac{1}{6} + \frac{5}{6^2} + \frac{5^2}{6^3} + \frac{5^3}{6^4} + \cdots =$

Neat Stuff

7. Repeat the Opener, except that Set Theory gets the first serve. What happens?

The players on Set Theory decided they weren't being paid enough . . .

8. Investigate the powers of the matrix

$$M = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}$$

Look for and explain some connections.

. . . so they formed a $\cup$.

9. **a.** If you haven't done Problem 8 on Day 4 yet, do it!
b. Create a matrix of transition probabilities for this situation.
c. Square your matrix. What up with that?

10. **a.** If you haven't done Problem 9 on Day 4 yet, do it!
b. Create a matrix of transition probabilities for this situation.
c. Raise this matrix to some powers. Wow! Why?

We'll do it live!

11. Carson is making two-flip sequences, looking for HH. The first two flips came up TT. He'll stop when he makes HH.

a. Create a matrix of transition probabilities for this situation.
b. What is the probability that Carson will achieve the goal of HH within 6 more flips?

Like Michael Jackson said, don't stop til you get enough heads.

12. For two pro volleyball teams, things change:

- when Tan Lines receives a serve, they win 90% of the points
- when Sin Waves receives a serve, they win 70% of the points

Tan Lines serves first. Repeat the Opener with this alternative information.

The winner faces Cos Play in the finals.

The players on Tan Lines love to touch things once then just keep moving away.

13. What proportion of the population lived in Continent A one year *before* Planet SA was discovered? Two years? Three years?

14. **a.** As the proportions for Continent S and Continent A change year to year, how far away are these proportions from the steady state?

b. Use this information to find a *closed form* (non-recursive) rule for $S(n)$, the proportion of the population in Continent S.

15. Calista is making three-flip sequences, looking for HTH. The first three flips came up TTT. She'll stop when she makes HTH.

a. Create a matrix of transition probabilities for this situation.

b. What is the probability that Calista will achieve the goal of HTH within 6 more flips?

16. Meaghan is negotiating to buy a used Toyota Matrix. She wants to pay $1500 for the car and thinks the seller is likely to want $2100. In the negotiation, each person is willing to meet the other halfway . . . repeatedly.

The seller, Neo Anderson, seems very eager to be rid of this car. He says he needs the money for kung-fu classes, but that seems unlikely.

a. What happens if Meaghan makes the first negotiating offer of $1800?

b. Is Meaghan better off making the first offer or waiting for the seller to talk first?

c. If the negotiating parties' prices are x and y, what eventual price will be reached?

17. Either build a transition diagram whose long-term proportions can have more than one steady state, or show that this is impossible.

Tough Stuff

18. Suppose A is a matrix of probability vectors such that for some power k, A^k is completely nonzero. Show that as n grows infinitely, the columns of A must all approach the same vector.

Looks like the problem set has reached a vertical Awesome-tote.