Day 6: Surfin' ABCDE

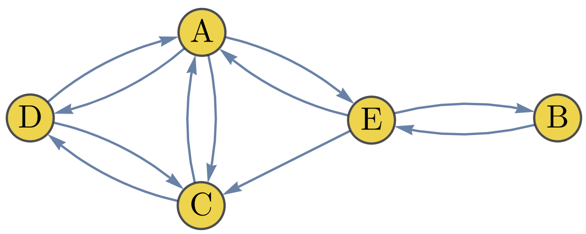
Opener

1. Now that all of her students have passed AP Stats, Amy is really bored and randomly surfing the web. However, she's still at school and the firewall only lets her visit these five web sites:

- AwesomeTotes.com (they sell bags)
- BenCTM.org
- CherryPi.com
- DayCart.com
- EulerHarmony.com

Doo doo da doo da doo . . . doo da doo da doo . . . doo doo da doo da doo . . . firewall.

Every minute, Amy randomly clicks a link to a web site. The sites are linked to each other in the following way:



Since A is only connected to C, D, and E, if Amy is currently at AwesomeTotes.com, she has a $\frac{1}{3}$ chance of clicking a link to either C, D, or E. Let $A(n), B(n), \ldots, E(n)$ be the probabilities that Amy is on each web site after n clicks.

a. Suppose Amy starts surfing at BenCTM.org. What is the probability that she will be at each of the five web sites after her first click? Her second click? Her third click? Please calculate this without using technology.

Click me, baby, one more time . . .

b. Create a 5×5 matrix T representing the transition probabilities for Amy moving from one site to another.

c. What expression produces the probabilities for Amy's surfing destination after 10 clicks, 20 clicks, 100 clicks? Use your favorite technology to calculate it.

d. How much do your answers to part c change if Amy started surfing at a different web site?

e. Imagine there are 1,000 teachers at Amy's school all surfing the web in this way. Which of the five web sites would you predict will be the most popular one, and why?

Did someone say *surfing*? Have you heard?

Important Stuff

2. Your matrix T from the Opener should have the property that every column sums to the same number. Explain why this must be true.

3. Juan gets a new puppy! He puts out a tank of water for the puppy to drink. By the time the puppy goes to sleep each day, only one-third of the water remains. Then Juan adds 1 liter at the end of the day. The tank contains 1 liter of water at the start of the first day.

A real one, not a video-game virtual pet like those Fibonaccis from the 90s.

a. The amount of water at the end of day n is related to the amount of water at the end of day $n-1$ according to this recursive equation:

$$W(n) = \boxed{} \cdot W(n-1) + \boxed{}$$

b. After many days, Juan observes that the amount of water in the tank at the end of each day is indistinguishable from the amount the day before; it has approached a *steady state*. At the steady state, describe how the amount of water in the tank changes throughout a day.

If you wanna be Juan's puppy, you have got to drink. Drinking is too easy, but that's the way it is.

c. Let x be the amount of water in the tank at the end of a steady state day. Write an equation you can use to find the value of x.

x, (the equation) gonna give it to ya.

d. Shira only sees the tank each morning before the puppy wakes, then at the end of each day after Juan has refilled it. From these observations, she only sees the amount of water increasing from day to day. How much water does she observe being added on the first day? Second day? Third day? Use these numbers to calculate the steady state amount of water.

The rest of the day, she is touring the facility and picking up slack.

4. **a.** Calculate the value of

$$1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots$$

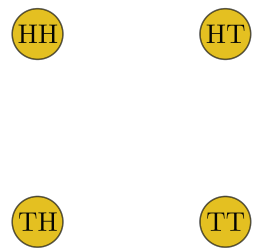
b. Calculate the value of

$$\frac{1}{4} + \frac{1}{12} + \frac{1}{36} + \frac{1}{108} + \cdots$$

Why'd you have to go and make things so complicated?

5. **a.** All four possible two-flip sequences of heads and tails are shown below. Add arrows to the diagram in the style of the Opener. Label the arrows with transition probabilities.

If you flipped HHTHTT, these sequences happened: HH→HT→TH→HT→TT.



b. Fill in the 4×4 matrix below with transition probabilities.

One, two, three four five, every number in this matrix's 0 or point five.

	from HH	from HT	from TH	from TT
to HH				
to HT				
to TH				
to TT				

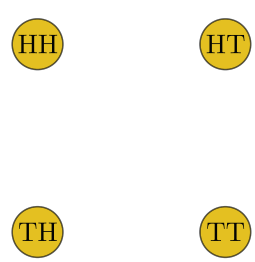
Remember that each column must sum to 1.

c. Square the matrix above. (Feel free to use tech.) Explain the result in the context of this problem.

6. Elizabeth and Elizabeth are playing a version of *Beat Cal* with two-flip sequences. They flip one coin repeatedly until either HH or TH comes up. If HH comes up, Elizabeth wins. If TH comes up, Elizabeth wins. In other words, both HH and TH act as "traps" since they end the game.

Which Elizabeth? We're not saying. Try the game a few times for yourself and see which Elizabeth wins!

a. Modify the diagram from the previous problem so that it describes this game. Label the arrows with transition probabilities.



b. Fill in the 4×4 matrix below with transition probabilities.

$$\begin{array}{c} \\ \text{to HH} \\ \text{to HT} \\ \text{to TH} \\ \text{to TT} \end{array} \begin{array}{c} \begin{array}{cccc} \text{from HH} & \text{from HT} & \text{from TH} & \text{from TT} \end{array} \\ \left[\begin{array}{cccc} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{array} \right] \end{array}$$

Remember that each column must sum to 1.

c. Let P be the matrix from part b. Use technology to calculate

$$P^{10} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

What does this result mean in the context of this problem?

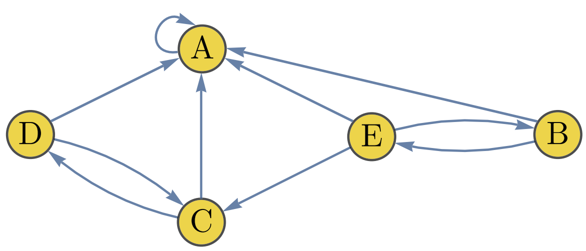
Why do you suppose the vector of repeated 1/4 was used?

Neat Stuff

7. If you haven't yet done Problems 4 and 5 from Day 5, please try them.

8. Repeat the Opener assuming that Amy's corner of the Internet looks like this instead:

Once Amy hits AwesomeTotes.com, it's bye bye bye to the rest of the world.



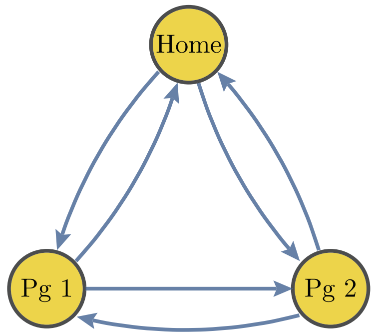
Suggest some ideas for how to modify the ideas in the Opener to measure the popularity of web sites to allow for sites like A that don't link outward.

9. Calculate the value of this sum.

$$S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \cdots$$

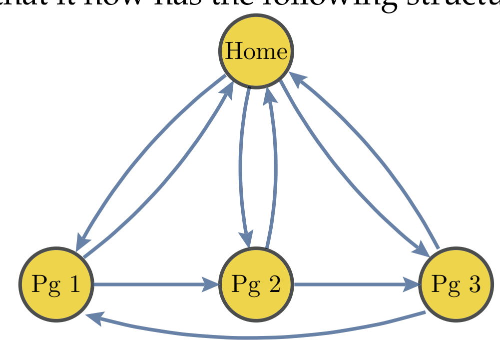
This sum has taken its toll on me, cause I can't use 1 over 1 minus x anymore. Whoa, oh, oh.

10. **a.** Stephen builds a web site about birds with this structure:



Assume that a surfer randomly clicks on a link every 10 seconds on Stephen's site in such a way that every outbound link has the same likelihood of being chosen. Use the ideas of the Opener to calculate the expected percentage of time that the surfer will spend on each page, over a long period of time.

b. Shireen adds an additional page to Stephen's web site so that it now has the following structure:



Calculate the expected percentage of time that the surfer will spend on each page, over a long period of time.

c. Generalize your results to a web site with a similar structure, but with n pages instead of 3.

11. **a.** Find a closed form rule (in terms of n) for all the elements of M^n given the matrix

$$M = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

So your website has three pages. That don't impress me much.

Did someone say *surfer*? Well, everybody's heard about Stephen's birds. Bird bird bird.

Oh my god, Becky. Page 3 got "back".

b. If n is large, determine the approximate value of

$$M^n \begin{bmatrix} .6 \\ .3 \\ .1 \end{bmatrix}$$

12. Here's an interesting matrix W:

$$W = \begin{bmatrix} \frac{1}{3} & 1 \\ 0 & 1 \end{bmatrix}$$

a. Compute $W^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $n = 1, 2, 3, 4$. What do you notice?

b. Compute $W^n \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ for $n = 1, 2, 3, 4$. What is different, and what is the same?

It's been . . . one week since the Iceland team, clapped their hands in the air to make a new meme . . . five days since they lost to France, they still got the rug burns on all their pants . . . it's been three days since the afternoon, McCallum spoke about Core in the teachers room . . . yesterday you found steady states, but it'll still be three days 'til our next problem date.

13. Ben drinks a can of TaB every 2 hours. Each can of TaB has 46.8 mg of caffeine, and 25% of the caffeine in Ben's body is gone by the time he drinks the next one. Ben starts with no caffeine and no TaB, then drinks his first TaB after 2 hours.

a. How much caffeine will Ben have in his system after 12 hours? Try to solve this problem using a matrix.

b. Find the steady state, if there is one.

c. Find a closed form for $C(t)$, the caffeine in Ben's system after drinking t cans of TaB.

Don't stop TaB drinkin' . . . hold on to that caffeine . . .

Tough Stuff

14. Find a matrix M for which M^8 is the 2×2 identity matrix, but none of M through M^7 is.

If you got here, you're going the distance. You're going for speed. Therefore you should re-read the norms!

15. Let $T(n)$ be defined by the rule $T(n) = nT(n-1) + n$ with $T(0) = 0$.

a. Find a closed form representation for $T(n)$.

b. Calculate this infinite product:

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{T(n)}\right).$$