Day 9: The Big Set Theory (vs Awesome-Totes)

Opener

1. Set Theory and Awesome-Totes are playing volleyball again. Both teams have improved; now Set Theory has a 75% chance to win each point, no matter who serves. They play a short game to 4 points, must win by 2. (Each rally is worth a point.)

   It's a tense game and the teams are tied 3-3.

   a. Find the probability that Set Theory wins the game on the next two points.

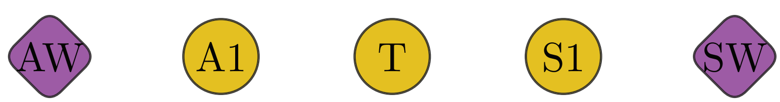
   b. Find the probability that Awesome-Totes wins the game on the next two points.

   c. Find the probability that after the next two points, the game is tied again.

The members of the original Awesome-Totes haven't been seen since last Friday, somehow.

2. There are 5 possible states the game can be in:
   - (**T**): the game is tied
   - (**S1**): Set Theory is ahead by one
   - (**A1**): Awesome-Totes is ahead by one
   - (**SW**): Set Theory wins by going ahead by two
   - (**AW**): Awesome-Totes wins by going ahead by two

   a. Complete this transition diagram for the game.



   b. Each person will use tools (coins, dice, the Wheel of Fish, etc.) to simulate this situation in Problem 1 twice. One person at your table should enter the number of wins for each team here:

   `http://bit.ly/pcmi2016`

The game can be in Utah, or Colorado, or Wyoming, or New Mexico, or Idaho. Oh wait, that's not what we mean at all, sorry.

Gosh, I hope these tools are appropriate and are being used strategically.

Important Stuff

3. **a.** What are the possible total amounts if you roll two fair dice, and what are their probabilities?

   b. What is the expected value of rolling two fair dice?

   c. What is the expected value of rolling one fair die?

A grid of options can work here. So can a tree diagram!

4. Multiply these. Avoid techmology if possible.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Techmology, what is that all about? Is it good, or is it wack? Is it good, is it wack, what is it all about?

5. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is sometimes called an *identity matrix*. Why might that be?

6. Hannah will spin the Wheel of Fish until she lands on the "10" space.

a. What is the expected number of spins she will need until she lands on the 10?

b. Hannah spins and lands on the 2. (Aww.) What is the expected number of spins that she will need from this point forward until she lands on the 10?

c. Hannah spins three more times and doesn't land on the 10. (Aww, aww, aww.) What is the expected number of spins that she will need from this point forward until she lands on the 10?

Hannah loves fish, or loves wheels, or something. She's spinning for the second day in a row!

7. Ayesha plays pickleball and has three matches scheduled. She is 70% likely to win against Brian, 80% likely to win against Brian, and 20% likely to win against Brian. What is the average number of matches Ayesha should expect to win?

OK, who just said "What's pickleball?" The first rule of pickleball is that you don't tell anyone what pickleball is.

Which Brian are we talking about? Who knows.

8. Define

$$X = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} \cdot \frac{5}{6} + 3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 + 4 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3 + \cdots$$

Show that

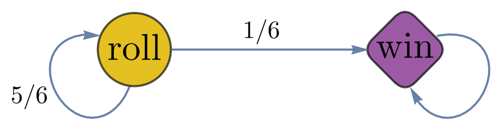
$$X = 1 \cdot \frac{1}{6} + (1 + X) \cdot \frac{5}{6}$$

9. What is the expected number of times that you will need to roll a die to get a 6? Prove it using Problem 8.

Neat Stuff

10. On average, how many turns will it take to reach the end space of this horribly simplistic board game?

$\frac{5}{6}$ of the time, the roll is wasted . . .



11. If you were to multiply

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 0 \\ 7 \end{bmatrix}$$

which of the $*$'s would you actually need to know to calculate the answer? Shade or color them in.

These problems have so many asterisks that we think we just broke a home run record.

12. If you were to multiply

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \left[\begin{array}{cc:ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix}$$

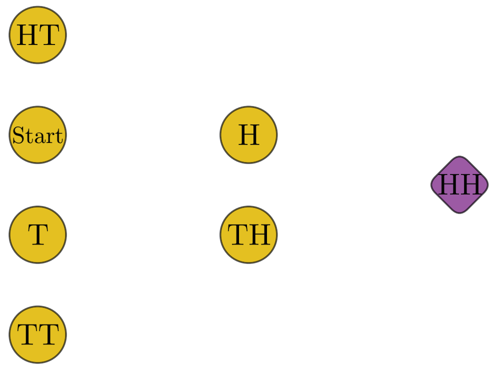
which of the $*$'s would you actually need to know to calculate the answer? Shade or color them in. Which of the ?'s do you know *have* to be zero? Which of the ?'s *could* be nonzero?

13. Heather is playing a game in which she flips a coin repeatedly until she gets HH.

We may have played this game a few times here.

a. This transition diagram shows the different states of the game. Connect these states with arrows labeled with probabilities.

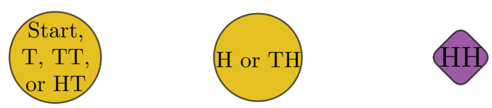
If she flips TTHH, then she goes through these states: Start→T→TT→TH→HH.



b. How can you be sure you didn't forget any arrows?

FORGOTTEN ARROW'D!!

14. **a.** Here's another version of the diagram from Heather's game. Connect these states with arrows labeled with probabilities.



b. Write down the transition matrix T for this game. Remember that each column must sum to 1.

c. Multiply this:

$$T^n \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{Put the 1 in the spot corresponding to your start space}$$

for $n = 1, 5, 10, 20$. What will the result approach as n keeps increasing?

d. Interpret the results in the context of the game.

15. Set Theory wins 75% of the points when playing Awesome-Totes. They play a short game to 4 points, must win by 2. What is the probability that Set Theory wins the game?

Note that as in the opener, if the game score gets to 3-3, it can continue for quite a while.

16. **a.** Find the standard deviation for the number of fish earned in one spin of the Wheel of Fish.

b. . . . two spins? three spins? 50 spins?

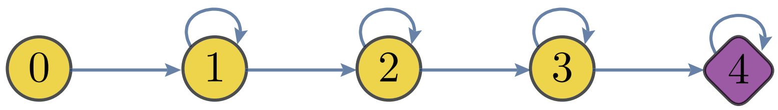
17. Anne wonders about the distribution of the number of fish on n spins of the Wheel of Fish.

Use techmology to build a histogram for the distribution of fish on two spins, three spins, 10 spins, and 50 spins. Notice anything?

Just from the context, this is clearly a Poisson distribution. Also, looks like techmology is good, and not wack!

18. Leslie will spin the Wheel of Fish until *all four numbers* are hit at least once. This transition diagram lists her progress:

She summons fish to the dish, she's living in a Chalet Swiss, she likes the sushi cuz it never touched a frying pan.



Label the transition diagram with the correct probabilities. Then, use what you've learned in the last two days to figure out the expected value of the number of spins Leslie will need to finish this job.

Gotta fish 'em all!

19. Gabi will roll a fair die until *all six numbers* are hit at least once. What is the average number of rolls it will take do to this?

Each number must pay for dinner when it is rolled.

20. Revisit today's Opener, but this time assume that Set Theory wins 90% of all rallies in which they serve and Awesome-Totes wins 70% of all rallies in which they serve. The team that wins a rally serves the next one. Set Theory is about to serve with the score tied.

21. A common misconception about squaring matrices is that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 \quad \text{is equivalent to} \quad \begin{bmatrix} a^2 & b^2 \\ c^2 & d^2 \end{bmatrix}.$$

Are there any situations in which this misconception happens to be correct? Generalize your work to larger square matrices.

A king-size bed is as close as it gets to squaring a matrice in real life.

22. At a particular high school, the standard math course sequence is Algebra 1, Geometry, Algebra 2, Precalc.

- All entering 9th graders start in Algebra 1.
- The pass rate for students is 80% in all classes.
- If students fail a class, they retake it next year.

This is a geometric sequence, but only for one term.

- Everyone takes math all four years. Everyone graduates. No one drops out.
- The same number of students that graduate each year enter as 9th graders the next year.

a. Draw a ten-state diagram to represent this situation. Since the same number of students graduate each year as enter, connect all the 12th grade states back to the Alg 1–9th state.

The states are Massachusetts, New Jersey, Maryland . . . alright alright, the states are Alg 1–9th, Alg 1–10th, Alg 1–11th, Alg 1–12th, Geo–10th, and so forth.

b. In the first year of the high school, there were 500 students in each grade taking their "on-time" course. What is the long-term distribution of students in each state? (Round to the nearest integer.) About how many students will be enrolled in each course in the long term?

Oh snap, there are more states in this transition matrix than there are in PARCC.

c. Assume that half the students have blue eyes, half have brown eyes. Because of systemic and institutional bias, students with blue eyes have a pass rate of 90% and students with brown eyes have a pass rate of 70%. Over the long term, what percentage of students in the Precalc class have brown eyes?

d. Think about some ways that you can extend or modify this problem. What questions are you interested in answering with this kind of model?

Extend this problem to Canada by replacing "9th grade" with "Grade 9".

23. Two teams play a series of games; the first team to win 4 games is declared the overall winner. Each team has an equal probability of winning each game. What is the expected number of games played?

Tough Stuff

24. Given transition matrix T and non-zero vector v so that $Tv = kv$ for some number k. Show that $|k| \leqslant 1$.

25. Figure this out.

Today's trig problem: just as insanely hard as the others! Yay!!

$$\cot\left(\frac{2\pi}{9}\right) + 2\cos\left(\frac{4\pi}{9}\right)\csc\left(\frac{2\pi}{9}\right) = \sqrt{3}$$

26. Find a fraction has the Fibonacci numbers in its decimal expansion:

$$0.001001002003005008013021034\ldots$$