Day 10: Wheel of Wheels!

Opener

1. Today we're playing *Wheel of Wheels*! Here's how to play:

- First spin this: `http://bit.ly/wheel-of-wheels`.
- If you land on the "8" space, you win 8 more fish immediately and you get to spin the Wheel of Wheels again! Hooray!
- If you land on "fish", "chips", or "peas", then you spin that corresponding Wheel once. You add however many fish you get to your current total and the game ends.

a. Each person should play this game twice. Record how many fish you won on each trial.

b. Combine your data with your tablemates. One person should enter the combined data here:

`http://bit.ly/pcmi2016`

ZOMG SO META

Unless you hate fish, then this is not a hooray moment.

The wheels on the websites go round and round, round and round, round and round . . . all through the Wi-Fi and/or LTE G4 connections.

Important Stuff

2. Jennifer is going on America's newest hit game show, *Big Boxes*! She has a 13% chance to hit the jackpot and win \$100,000. But if not, she still takes home a consolation prize of \$1,000.

Write the expected value of Jennifer's payout by filling in some fancy boxes!

$$\mathsf{EV} = \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} = \boxed{}$$

3. **a.** Let F be the expected number of fish won from one game of the Wheel of Wheels. Suppose Michael lands on the "8" on his first spin. He is about to spin again. At that moment, what is the expected number of *additional* fish that he will win? What is the expected number of *total* fish that he will win?

b. Determine the value of F using this equation:

$$\mathsf{F} = \underbrace{\boxed{} \cdot \frac{1}{4}}_{\text{fish}} + \underbrace{\boxed{} \cdot \frac{1}{4}}_{\text{chips}} + \underbrace{\boxed{} \cdot \frac{1}{4}}_{\text{peas}} + \underbrace{\boxed{} \cdot \frac{1}{4}}_{\text{8 \& spin again}}$$

These wheels are fun, but the Wheel of Morality is still the best wheel ever.

Amazingly, this is the same skill Jennifer will need to succeed on *Big Boxes*!

One of these fish may have saved Pittsburgh.

Psst: the answers here might involve variables!

4. In Day 9's Opener you built a transition diagram for the volleyball game (tied at 3-3).

a. Build a 5-by-5 transition matrix T for the game. While you don't have to, it may be helpful to list the five states in the order given by the transition diagram.

b. Use techmology to calculate T^2 and interpret the numbers you see. What parts of this matrix are you most interested in?

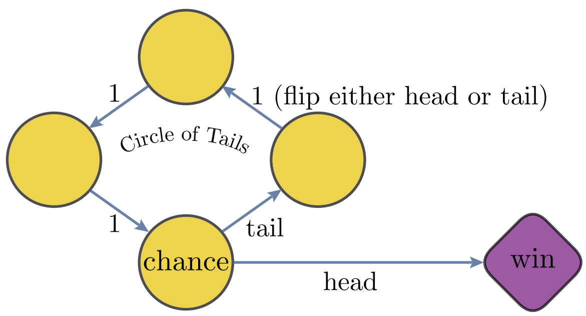
c. What's the probability that Set Theory wins?

Remember, Set Theory has a 75% chance to win each point.
Keep reading the sidenotes until the back page; we've got a surprising problem about PIN numbers!

5. In a different volleyball game, Awesome-Totes leads 4-3. What's the probability that Awesome-Totes wins?

Fun fact: even though the names Set Theory and Awesome-Totes were selected by teachers and students, they serve our purposes of including more STAT in the course, and also more references to the AT-ST from Return of the Jedi.

6. Dylan uses a fair coin to play the *Circle of Tails*!



The game starts at the "chance" space. If Dylan flips heads at the "chance" space, he wins! But, if he flips tails at the "chance" he has to go through the Circle of Tails before he can try again. He wants to figure out how many flips it will take to complete this game, on average.

It's the Circle of Tails! It doesn't move us all. It's not the wheel of fortune. It's not even a leap of faith. It's a path of sadness. But: things can only get better. Whoa, oh, oh oh oh.

a. Let D be the expected number of flips it takes Dylan to complete this game. Suppose Dylan flips tails on his first turn and goes down the entire Circle of Tails. He returns to the "chance" space, and is about to flip again. At that moment what is the expected number of turns for him to complete this game *from that point on*? Including his tour of the Circle of Tails, how many turns is it *in total*?

b. Determine D using this equation:

Hannah waves at Dylan at the end of the Circle of Tails, but is happier because she could try again right away.

$$D = \boxed{} \cdot \frac{1}{2} + \boxed{} \cdot \frac{1}{2}$$

7. Fill in the missing blanks. Explain what was done to go from one line to the next.

$$\bigstar \quad D = 1 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} + 9 \cdot \frac{1}{8} + 13 \cdot \frac{1}{16} + \cdots$$

$$D = 1 \cdot \frac{1}{2} + \frac{1}{2}\left(5 \cdot \frac{1}{2} + 9 \cdot \frac{1}{4} + 13 \cdot \frac{1}{8} + \cdots\right)$$

$$D = 1 \cdot \frac{1}{2} + \frac{1}{2}\left[\underbrace{\left(1 \cdot \frac{1}{2} + 5 \cdot \frac{1}{4} + 9 \cdot \frac{1}{8} + \cdots\right)} + \underbrace{\left(\square \cdot \frac{1}{2} + \square \cdot \frac{1}{4} + \square \cdot \frac{1}{8} + \cdots\right)}\right]$$

$$D = 1 \cdot \frac{1}{2} + \frac{1}{2}\left[\qquad \square \qquad + \qquad \square \qquad\right]$$

Use the work above to calculate the value of D.

8. Explain the connection between Problems 6 & 7. Then, explain the significance of the numbers in the equation marked with ★.

Your explanation to this problem should *not* involve cauliflower chicken and waffles. Under no circumstances should this happen. Don't get weird.

9. Calculate each result.

a. $\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 0 \\ -3 & -2 & 3 \\ -2 & -1 & 1 \end{bmatrix}$

10. Use techmology to calculate this, an *inverse matrix*:

$$\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix}^{-1}$$

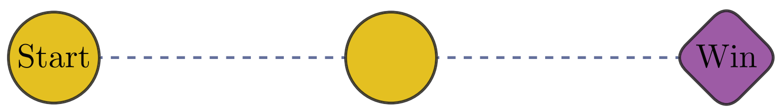
If using a TI-Nspire, use [^] to "raise" to the −1 power. (Remember to use [(−)] for negative, not minus sign, when entering in −1.)

Makes me wonder what $\begin{bmatrix} 1 & 1 & -3 \\ -3 & -2 & 6 \\ -1 & 0 & 1 \end{bmatrix}^0$ is. Oh, cool!

Neat Stuff

11. Let's revisit Cesar's game from Problem 7 on Day 2. Cesar starts in the circle marked "Start" below. He flips a coin as many times as he wants. When he flips a head, he moves forward one space. When he flips a tail, he moves back one space (if possible). If he reaches the "Win" space, he wins!

Near! . . .

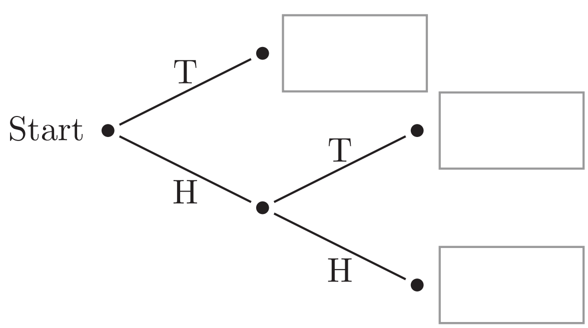


Now we'll calculate the expected number of flips until Cesar wins.

a. Complete the transition diagram above.

b. Here is a partial tree diagram of what could happen when Cesar starts the game. In the blank boxes, write the state that Cesar would be in, given the specific coin flips. For each box, also determine the probability that Cesar's flips would lead him to that box.

A quick salute to Utah trees. OK, moving on.



These are *Big Boxes*! Unlike the box Becky won on *Wheel of Fish*, when you're done they won't be empty.

c. Let W be the expected number of flips until Cesar reaches the "Win" space. Determine the value of W using this equation:

$$W = \square \cdot \square + \square \cdot \square + \square \cdot \square$$

d. Explain the connection between your answer for the expected number of flips in Cesar's game with the data we collected on Day 7 at

`http://bit.ly/pcmi2016`

. . . Far!

12. Solve for any variables in each problem.

Near! . . .

a. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ -3 \\ -12 \end{bmatrix}$$

c. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 15 \end{bmatrix}$$

d. $$\begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix} + \begin{bmatrix} -12 \\ -3 \\ -12 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 15 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Vectors and matrices add like you think they should: cell by cell.

e. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

f. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

13. Multiply these any way you like:

a. $$\begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 & -2 \\ 3 & -5 & -2 \\ -5 & 8 & 3 \end{bmatrix}$$

b. $$\begin{bmatrix} 3 & -4 & -2 \\ 3 & -5 & -2 \\ -5 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & -2 \\ 1 & -1 & 0 \\ -1 & -4 & -3 \end{bmatrix}$$

14. Leah and Mo are playing a game using dice. They roll two dice; if the sum is 5, Leah wins. If the sum is 10, Mo wins. If the sum is anything else, they re-roll. Find the probability that Leah wins, along with an explanation that doesn't involve anything infinite.

If the sum is 15, they'll send the dice back to the manufacturer.

. . . Far!

15. Team Mystic and Team Valor are playing a game to 4 points, must win by 2. Team Mystic wins points with probability p. The score is 3-3. What is the probability that Team Mystic wins the game?

These are really the only two teams, right? Heck yeah. Team Instinct, boo!

16. In the Utah "Pick-4" lotto you can play a ticket that wins under any order of the numbers you play. If you play 2357, you'd also win with 5372 or 7352. All those tickets are the same. How many *different* tickets are possible for this lotto?

2357? Hey, isn't that your PIN number?

We're kidding, of course. Utah doesn't have a lotto! Neither does Alaska, Hawaii, Mississippi, Alabama . . . and Nevada.

17. Find the number of different non-negative integer solutions to the equation

$$a + b + c + d + e + f + g + h + i + j = 4$$

18. Alex, Angelina, Chris, Danilsa, Deva, and Manish go to dinner every night and play "credit card roulette" to randomly decide who pays.

a. Find the expected value for the number of meals it will take until each of them has paid at least once.

b. Is it possible, for some larger $n > 6$, for the expected number of meals to be larger than $3n$? If so, find the smallest n for which this happens. If not, explain why.

If k people have paid already, what's the probability someone new will pay for the next meal? What's the expected value of the number of meals it will take until someone new pays?

Tough Stuff

19. Mystic and Valor play to 4 points, must win by 2. Mystic wins points with probability p. The score is 0-0. What is the probability that Mystic wins the game?

20. Show that the expected number of turns needed to flip HTH is eight.

21. Figure this out.

$$8\cos\left(\frac{2\pi}{13}\right) + 2\left(1-\sqrt{13}\right)\cos\left(\frac{4\pi}{13}\right) + 4\cos\left(\frac{6\pi}{13}\right)$$
$$= 1 + \sqrt{13}$$

Near . . . far . . . wherever you are, I believe that this trig will go on . . . Once more, you open your set. And it's here in Tough Stuff, and this trig will go on and on . . .