Day 11: Dice Dice Baby

Opener

1. Today we're playing *Dice Dice Baby*! Here's how to play:

Yo D&B, let's kick it!

- Roll a die.
- If you get a 1, you win \$1 and the game starts over again.
- If you get a 2, you win \$2 and the game starts over again. (Same for 3, 4, and 5—you win what the die says.)
- You accumulate all winnings in this game.
- But, if you get a 6, you lose \$16 and the game ends.
- You must keep rolling until you roll a 6.

Keep rollin', rollin', rollin' . . .

a. Each person should play this game twice. Record your total earnings, positive or negative, on each trial.

b. Combine your data with your tablemates. One person should enter the combined data here:

`http://bit.ly/pcmi2016`

Important Stuff

2. A family has four children. One is 4 years old, one is 6 years old, one is 3 years old, and one is eight years older than the average age of the four children. What is the average age of the four children?

It's Friday! Each child's gotta be fresh, gotta go downstairs, gotta have cereal.

3. Define D as the expected total earnings of one game of *Dice Dice Baby*.

a. Sam starts the game by rolling a 5. She is about to roll again. At that moment, what is her expected total earnings (in terms of D)?

I'm on a roll, and it's time to go solo, rollin a 5.0.

b. Write an expression for D. You may need more boxes:

$$D = \boxed{} \cdot \frac{1}{6} + \boxed{} \cdot \frac{1}{6} +$$

c. Solve for D, and compare your value of D with the class average.

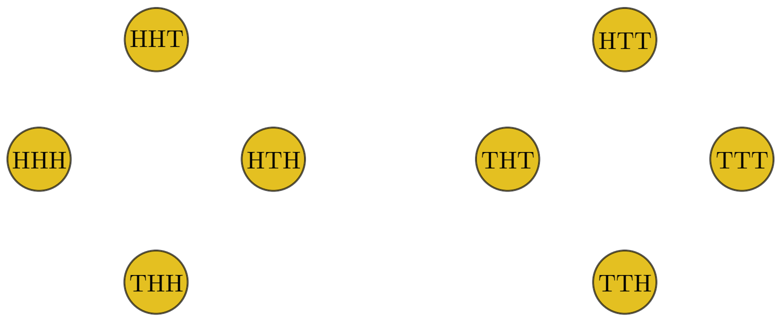
If there was a problem, yo, you solved it.

4. Suppose Becky flips a coin repeatedly and keeps track of the three-flip sequences that occur.

a. Model this situation by connecting the eight three-flip sequences in the transition diagram below with arrows.

Becky needs to break away! If Becky flips TTH-HTHTTT . . . then her states are TTH→THH→HHT→HTH and so on.

Darryl suggests one color for H flips and another color for T flips. It's cool!



How do you know you didn't forget any arrows?

b. Write down the transition matrix T for this Markov chain.

Markov! Pole-ov. For more Markov madness this weekend, try reading a few "Calvin and Markov" strips.

from

to	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
HHH								
HHT								
HTH								
HTT								
THH			$\frac{1}{2}$	0				
THT			$\frac{1}{2}$	0				
TTH			0	$\frac{1}{2}$				
TTT			0	$\frac{1}{2}$				

Remember that your columns should sum to 1.

c. Multiply this:

$$T \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{bmatrix}$$

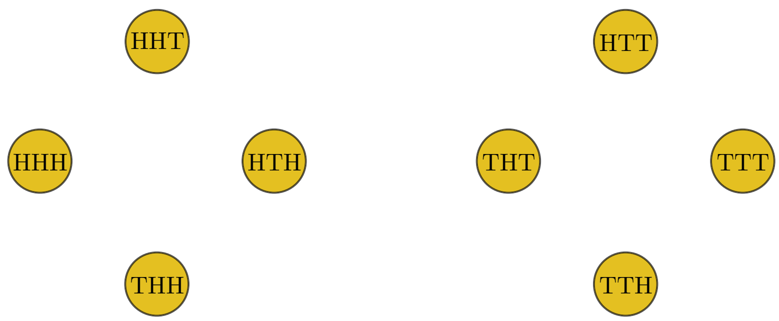
This vector's going down, though it doesn't seem to be yelling anything.

What does the result mean in this context?

5. In the *Beat Cal* game, two players pick a three-flip sequence. Then one coin is flipped repeatedly until one of the two sequences appears; that player wins. Suppose Cal picks HHH and you pick HTT.

Beat Cal, beat Cal . . . say that you'll beat Cal . . . go on and beat Cal . . . I can't care bout anything but flips.

a. Model this situation by connecting the eight three-flip sequences in the transition diagram below with arrows.



b. Write down the transition matrix T for this Markov chain.

And I would walk 500 miles, and I would walk 500 more, just to be the man who walked a thousand miles to hatch this darn Pokémon egg.

from

to	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
HHH								
HHT								
HTH								
HTT			0	1				
THH			$\frac{1}{2}$	0				
THT			$\frac{1}{2}$	0				
TTH			0	0				
TTT			0	0				

Remember that your columns should sum to 1.

c. Multiply

$$T^n \begin{bmatrix} \frac{1}{8} \\ \vdots \\ \frac{1}{8} \end{bmatrix}$$

That vector has eight $1/8$s in it. We were just too lazy to write them all. And by "lazy" we mean "hopeful that we could fit all this on one page".

for $n = 10, 20, 50$. What does the result mean in this context?

This problem's result is so good, so good, so good!

d. Compare your result with the data from Day 3 at

`http://bit.ly/pcmi2016`

6. Hannah continues to spin the Wheel of Fish hoping for a ten. Her status after n spins can be modeled by this

We just might have a problem that you'll understand. We all need somebody to solve this.

function:

$$P(n) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & \text{if } n = 0 \\ \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & 1 \end{bmatrix} P(n-1), & \text{if } n > 0 \end{cases}$$

a. Determine $P(0)$, $P(1)$, $P(2)$, and $P(3)$ by hand to figure out what this function is saying.
b. For $P(3)$, describe as precisely as you can what the two numbers refer to.
c. Based on your calculations, write a formula for both numbers in $P(n)$ as a function of n.
d. What do you get if you add *all* the top numbers of each $P(n)$? What does this value represent?

Refer back to Day 7 for more context, and a reminder that we introduced the Wheel of Fish on Monday. I know, it feels like it's been in your life forever.

Neat Stuff

7. **a.** Repeat Problem 5 but for THH vs TTH. What is the likelihood that each side wins?
b. Repeat for TTH vs HTT.
c. Repeat for HTT vs HHT.
d. Repeat for HTT vs THH.
e. What do the above results say about what three-flip sequences are best in the *Beat Cal* game?

This problem is a great opportunity for your table to divvy up this work. Otherwise, you might work on this all night long (all night).

8. Set Theory and Awesome-Totes are tied at 3-3, must win by 2. Set Theory has a $\frac{3}{4}$ probability to win each point.

a. What is the probability that the game ends in exactly 2 points?
b. . . . in exactly 3 points? 4 points?
c. If the expected number of points needed is N, show that

$$N = \boxed{} \cdot 2 + \boxed{} \cdot \boxed{}$$

d. Determine the value of N.

Under no circumstances does N stand for Nickelback. That is unacceptable.

9. On Day 10 you built a transition matrix T for the volleyball game. A vector v with a 1 and four zeros is a probability vector that says the game starts out tied.

a. Calculate Tv. Then calculate $v + Tv$. What do these numbers tell you?

The 1 in v will match whatever state you associated with a tie.

b. Calculate T^2v. Then calculate $v + Tv + T^2v$. What do these numbers tell you? (They can't be used as probabilities.)

From here on, we'll use Kiendra's suggestion and refer to this as "that thing".

c. Calculate T^3v. Then calculate $v + Tv + T^2v + T^3v$.

d. Calculate $v+Tv+T^2v+T^3v+\cdots+T^7v$. Compare the sum of the three smallest numbers in this vector to the results of Problem 8. What's going on?

This looks like a job for tech. It beckons you to come to its window.

10. Suppose Irene wants to play *Beat Cal* against you and chooses one of the three-flip sequences. Which other sequence should you choose to maximize your chances of winning?

Come on, Irene!

11. In "side-out" scoring for badminton and some other games, only a serving team can win a point; the other team can win the serve but not a point.

But not pickleball, duh.

a. Two teams each win 50% of their rallies. What is the probability that the serving team wins the next point? (It's not 0.5.)

b. If the initially serving team has probability p of winning each rally, what is their probability of winning the next point?

Possibly misheard lyrics from last night's karaoke: "I'm a sailor peg, and I lost my hammer! Climbing up the topsails, I lost my hammer!"

12. Problem 9 asked you to calculate

$$S = \sum_{n=0}^{7} T^n v = v + Tv + T^2v + T^3v + \cdots + T^7v$$

There wasn't really any reason to stop at 7. So, what about

$$\begin{aligned} S = \sum_{n=0}^{\infty} T^n v &= v + Tv + T^2v + T^3v + \cdots \\ &= (I + T + T^2 + T^3 + \cdots)v \end{aligned}$$

Here, I is an identity matrix that is the same size as T, with ones on the diagonal. In this case I is short for I *WANT* IT THAT *WAY* screamed at high volume.

a. Say, this looks a lot like a kind of thing we saw earlier in the course. What thing? Under what circumstances do those things "converge" to an answer?

b. Is there anything about matrix T that could make this *not* converge? If you're not sure, try calculating the sum for a large number of terms.

c. What if you used a smaller matrix made of just some of the states of T? Which states? Play around and see what happens.

13. Die 1 has faces 3, 3, 3, 4, 4, 4. Die 2 has 2, 2, 2, 2, 6, 6. Die 3 has 1, 1, 5, 5, 5, 5. Robert will allow you pick any die. Then he'll select one of the remaining dice. You'll both roll, best number wins. What die should you pick?

Say it ain't so! Your choice is a heartbreaker, Robert.

14. In Problem 5 you multiplied T^n by a vector of $\frac{1}{8}$. Ignore that and focus on T^n itself where n is a high power. Each column of T^n gives you some information . . . what information? Does this change the fact that we said HTT beats HHH with a specific probability?

15. A company is hoping to hire the best of 6 possible candidates for a job. After seeing each candidate they must make an immediate yes-or-no decision about whether to accept that candidate, without knowing who else is coming. What strategy should the company use to maximize its chances of hiring the best of the 6 candidates?

Tough Stuff

16. Risky players can try *Exploding Dice Dice Baby*. The 6 now pays \$6 and continues the game. But any rolled number is set to explode if it is hit again. Players can stop anytime they want and take their money, but if they make all six numbers without exploding they win a \$20 bonus. Exploding doesn't cost money but kills all winnings.

Determine the expected value of playing this game, assuming the player makes decisions that are always in the interest of increasing EV.

Word. Everyone enjoy your weekend among lonely goats and herders! Yo de lay heeeee . . . yo de lay heeeeee . . . yo da loo.

Just like on *Boom!*. Except these explosions have less Kraft Dinner and more dijon ketchup.

17. We were able to draw all 8 three-flip sequences and their connections in a planar graph (i.e., no crossings). Draw all 16 four-flip sequences and their connections in a planar graph, or show that this is impossible.

18. Figure this out.

$$\cot\left(\frac{2\pi}{15}\right) + 4\sin\left(\frac{2\pi}{15}\right) = \sqrt{15}$$

Even more impossible triggy things!

. . . especially if they shoot explosive farts! Good job by Smokehaus, using a Ponyta as its logo! Ponytas are so pretty! They come from La Isla Ponyta.