Day 12: Flippy Bird

Opener

1. Today we're playing Cesar's game, also known as the "flip HH" game. The game starts in the 0 state. Cesar flips a coin repeatedly. Flipping H moves Cesar to the right one state. Flipping T moves Cesar to the left one state (if possible). The game is over when Cesar reaches the 2 state.

Flip HH is also the name of a new show on the History Channel where they flip haunted houses for profit.

a. Draw arrows connecting the states in the transition diagram above. Label each arrow with its transition probability.

b. Each person will play Cesar's Game three times, keeping track of their states <u>and</u> flips during the game using the table on the back of this page.

I liked the Cesar's Game book better than the movie, but the movie was alright.

c. Fill out the other table on the back. Combine the data at your table. One person should enter the combined data here:

`http://bit.ly/pcmi2016`

Important Stuff

2. Find a simple expression for this sum:

$$S = a + aR + aR^2 + aR^3 + \cdots$$

where $|R| < 1$.

What's a pirate's favorite geometric series? $aR + aRR + aRRR + aRRRR + aRRRRR + aRRRRRR \cdots$

3. Mary will flip a coin that she somehow comes up heads with probability p.

a. If Mary flips this coin 10 times, how many of those flips should she expect to come up heads?

b. What is the expected number of flips Mary will need to perform before she flips a head?

This is one of my favorite places to visit in Arizona.

4. Based on the class data from the Opener, on average, about how many flips does it take to complete Cesar's game? How many flips come from the 0 state? How many from the 1 state?

Once you reach the 2 state, you can stop flipping coins and fill in 2 all the way down.

Game 1		Game 2	Game 3
0	Initial state	0	0
	Flip #1		
	State #1		
	Flip #2		
	State #2		
	Flip #3		
	State #3		
	Flip #4		
	State #4		
	Flip #5		
	State #5		
	Flip #6		
	State #6		
	Flip #7		
	State #7		
	Flip #8		
	State #8		
	Flip #9		
	State #9		
	Flip #10		
	State #10		
	Flip #11		
	State #11		
	Flip #12		
	State #12		
	Flip #13		
	State #13		
	Flip #14		
	State #14		

# of your games for which...	
flip #1 was made while in the 0 state	3
flip #2 was made while in the 0 state	
flip #3 was made while in the 0 state	
flip #4 was made while in the 0 state	
flip #5 was made while in the 0 state	

# of your games for which...	
flip #1 was made while in the 1 state	0
flip #2 was made while in the 1 state	
flip #3 was made while in the 1 state	
flip #4 was made while in the 1 state	
flip #5 was made while in the 1 state	

5. Every night at PCMI, Amy, Brian, Chris, and Danny go out to dinner and play "credit card roulette": the waiter picks one of their credit cards at random (with equal probability) to pay for the meal.

A strange game. The only winning move is not to play!

a. Use dice or the Wheel of Fish to simulate what happens each night until every person has paid at least once. Fill in the chart below with what happens each night.

Dinner #	Who paid?	# people who have paid at least once
1		1 person
2		
3		
4		
⋮		

b. Each person at your table will simulate this situation above two more times, then fill in this chart:

Don't think *too* hard about how many nights it takes until at least 1 person has paid. I suppose with credit cards, the real answer is "you'll pay eventually . . . a lot".

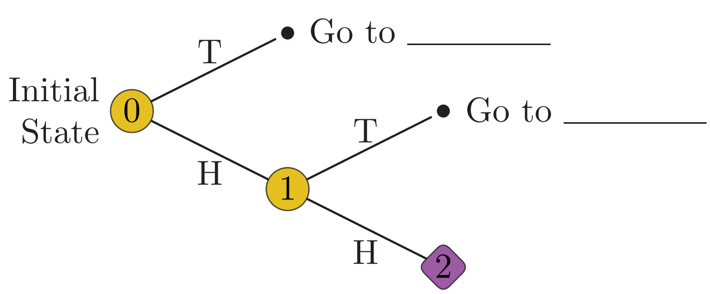
Across 3 simulations, the # of nights until . . .
1 person paid at least once =
2 people paid at least once =
3 people paid at least once =
Everyone paid at least once =

c. Combine your data at your table. One person should enter the combined data here on the tab marked "Day 12 CCR":

`http://bit.ly/pcmi2016`

6. **a.** Here is a partial tree diagram of what could happen in Cesar's game. Fill in the blanks with what happens in each situation.

If you have notes from Problem 11 on Day 10 (Neat Stuff) they may be helpful here.



b. Let W be the expected number of flips until Cesar reaches the 2 state. Determine the value of W using this equation:

$$W = \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{} + \boxed{} \cdot \boxed{}$$

7. **a.** Write a 3-by-3 transition matrix T for Cesar's game, using the state order provided in the opener.

b. With $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, compute by hand the sequence

$$v, Tv, T^2v, T^3v, T^4v, T^5v$$

Computing by hand is still better than computing by foot . . .

Use fractions instead of decimals, and write all fractions within each vector using a common denominator.

Also, hop up and down on one leg while reciting the alphabet backwards.

c. Look for patterns: figure out what T^6v is without matrix multiplication, and describe a general term for the first element of these vectors.

8. **a.** How do your results from Problem 7 compare to the empirical data found in the shared spreadsheet?

b. How could you use information from Problem 7 to determine the average number of flips from the 0 state? From the 1 state? The total number of flips?

Review Your Stuff

9. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **<u>one</u>** review question for tomorrow's problem set. Spend **<u>at most 15 minutes</u>** on this. Make sure your question is something that ∗**everyone**∗ at your table can do, and that you expect ∗**everyone**∗ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. You can do it! And so can everyone!

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff

10. Fill in the missing blanks. Explain what was done to go from one line to the next.

$$
\begin{aligned}
S &= 1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \frac{8}{32} + \cdots \\
S &= 1 + \frac{1}{2}\left(1 + \frac{2}{2} + \frac{3}{4} + \frac{5}{8} + \frac{8}{16} + \cdots\right) \\
S &= 1 + \underbrace{\frac{1}{2}\left(1 + \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{5}{16} + \cdots\right)} + \frac{1}{2}\left(\qquad\qquad\qquad\right) \\
S &= 1 + \qquad\quad \downarrow \qquad\quad + \underbrace{\frac{1}{4}\left(\qquad\qquad\qquad\right)} \\
S &= 1 + \qquad \boxed{} \qquad + \qquad \boxed{}
\end{aligned}
$$

Use the work above to calculate the value of S.

11. Recall v, Tv, $T^2v, \ldots$ for Cesar's game from Problem 7.

Recall, but don't have Total Recall. You've been at PCMI for *two weeks* . . . two weeks . . .

a. Find the sum of *all* first elements in the vectors from Problem 7:

$$v + Tv + T^2v + T^3v + T^4v + \cdots =$$

Interpret this number in the context of Cesar's game.

b. Find the sum of all second elements in that same vector. Interpret that number in this context.

c. What does the third element in these vectors represent? Explain why the third elements have an infinite sum.

The sum of the fifth element of these vectors is Milla Jovovich.

12. Find a simple expression for this sum, where T is a matrix and v is a vector:

$$S = v + Tv + T^2v + T^3v + \cdots$$

If we go by Problem 2 then this also needs $|T| < 1$? What does that even mean for matrices?

13. If you didn't get the chance, try Problem 8 from Day 11.

14. Here are three Texas Hold'em hands:

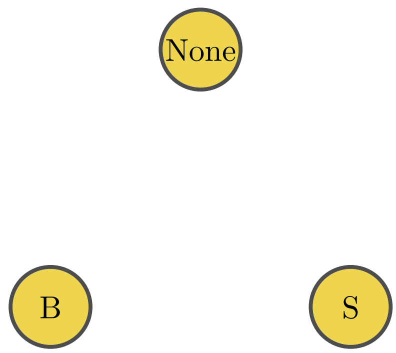
- Hand 1: jack of spades, ten of spades
- Hand 2: ace of diamonds, king of hearts
- Hand 3: two of diamonds, two of hearts

There are several "calculators" online that will determine the percentages for poker hands. Guess at your intuition about this problem, then use a poker calculator to check.

Robert will allow you to pick any hand. Then he'll select one of the remaining hands. You'll both go all in, best hand wins (after the flop, turn, and river). What hand should you pick?

15. Here is a record of the meat offerings at breakfast over the last few weeks at Zermatt.

Day	Meat
July 1	Bacon
July 2	Sausage
July 5	None
July 6	None
July 7	None
July 8	None
July 11	Sausage
July 12	Bacon
July 13	None
July 14	None
July 15	Sausage
July 18	None



We have chosen to ignore the other meat offering at breakfast: on two days last week, Zermatt offered "Assorted Beagles with Cream Cheese". They claimed that was vegetarian; perhaps the beagles were themselves vegetarian. Truly a sordid tale of assorted tails.

The meat offering at breakfast can only be one of three things: bacon, sausage, or no meat. The meat offering at breakfast is chosen randomly each day and that the probability of what will be chosen next depends only on what was chosen the previous day.

For those who like bacon and sausage, the top state should be relabeled with the letters from the bottom states.

a. Use the data above to create a Markov chain model for the meat offering at breakfast.

b. Based on this model, what could we expect for breakfast tomorrow and with what probabilities?

c. If this model is accurate and Zermatt continues offering breakfasts in this manner for a long time, what percentage of breakfasts will feature bacon? Sausage? No meat?

16. The BACN assessment is adaptive. All questions are categorized as either easy, medium, or hard. If a student gets two easy questions in a row right, then she gets a medium question next. If a student gets two medium questions in a row right, then she gets a hard ques-

You can tell a student is done with the BACN when their brain starts to sizzle.

tion next. If a student gets a medium or hard question wrong, then her next question will be one level easier. The assessment always starts with an easy question.

“If [[NAME]] has 3 strips of bacon and [[NAME]] has 5 strips of bacon, what is the area of their bacon in square strips?”

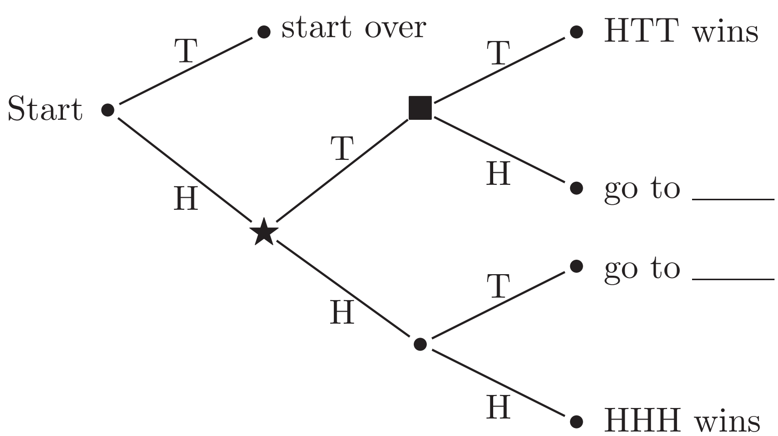
Monica is taking the BACN assessment. She has a 90% chance of getting each easy question right, a 75% chance of getting each medium question right, and a 40% chance of getting each hard question right.

a. Create a Markov chain to model this situation. How many states should there be?

b. After Monica answers many questions, approximately what proportion of easy, medium, and hard questions will she encounter? What proportion of all questions will she get right?

17. Danny and Danny are playing *Beat Cal*. Danny chooses HHH and Danny chooses HTT. Here is a partial tree diagram showing the different possible outcomes. Fill in the two missing blanks with either ★ or ■.

As usual, which Danny we’re talking about, we won’t say. Wait, shouldn’t this be called *Beat Danny*, then? Nah, that would be confusing.



Define X as the expected length of the entire game, Y as the expected length of the game from the ★ and Z as the expected length of the game from the ■. Complete these three equations and use them to solve for X.

There are other ways to find the expected length of the game, but it’s pretty amazing how well this works!

$$X = \square \cdot \frac{1}{2} + \square \cdot \frac{1}{2}$$

$$Y = \square \cdot \frac{1}{4} + \square \cdot \frac{1}{4} + \square \cdot \frac{1}{4} + \square \cdot \frac{1}{4}$$

$$Z = \square \cdot \frac{1}{2} + \square \cdot \frac{1}{2}$$

18. Determine the expected number of flips until a *Beat Cal* game involving HHT and HTH ends.

19. Cal is tired of losing so many games of *Beat Cal*. He is secretly going to create an unfair coin that produces heads with probability p to use against his next opponent. With a fair coin, HTT will win against HHH 60% of the time. What value of p would be required so that HHH will win against HTT 50% of the time?

Perhaps he shouldn't play a game which is titled *Beat Cal*, then. It's really pretty obvious.

It's a testament to Cal that even when given the opportunity to cheat, he cheats only just enough to make things fair again.

20. During Day 11 we noticed there was a very high range of wins and losses while playing *Dice Dice Baby*. One way to measure this is to use *variance*. Variance is a short name for *mean squared deviation*, which is how to calculate it:

- For each result, find its *deviation*, its distance from the mean.
- Take the deviations and *square* them.
- The variance is the *mean* of the squared deviations.
- The square root of variance is the *standard deviation*. Its use is more popular because it is in the same units as the original data.

a. Find the variance of the number of fish won in one spin of the Wheel of Fish.

b. . . . the Wheel of Wheels.

c. . . . one play of *Dice Dice Baby*, assuming we'd pay you in fish, which we totally would.

Actually, standard deviation is popular just by its name, like the American Standard brand of toilet.

Tough Stuff

21. After 16 coin flips, Vicki notices there were 11 heads and 5 tails. What is the probability that, throughout the entire flipping experience, "heads" was ahead in the count? (1-0, 2-0, etc.) Generalize to n flips.

22. Calculate $\displaystyle\sum_{n=0}^{\infty} \frac{1}{F(2^n)} = \frac{1}{F(1)} + \frac{1}{F(2)} + \frac{1}{F(4)} + \frac{1}{F(8)} + \cdots$.

The Fibonacci sequence $F(n)$ is rabbitly defined as $F(0) = 0$, $F(1) = 1$, then each $F(n) = F(n-1) + F(n-2)$.

23. Yay! More obnoxious trigonometry stuff to figure out!

$$4\cos\left(\frac{6\pi}{21}\right) - 2\left(1+\sqrt{21}\right)\cos\left(\frac{4\pi}{21}\right) + 2\left(5+\sqrt{21}\right)\cos\left(\frac{2\pi}{21}\right) = 7+\sqrt{21}$$