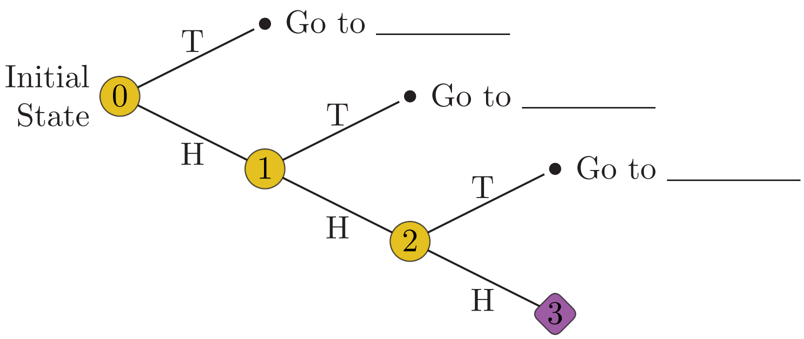
Day 13: Flip Out

Important Stuff

1. Today's exciting game is the Flip-HHH-Game! Keep flipping a coin until you get three heads in a row.

Triple H is here! The Game has arrived!

a. Complete this partial tree diagram.



b. What is the expected number of flips until HHH?

c. Compare your answer with Day 2's class average:

`http://bit.ly/pcmi2016`

2. Every night at PCMI, Amy, Brian, Chris, and Danny go out to dinner and play "credit card roulette": the waiter picks one of their credit cards at random (with equal probability) to pay for the meal.

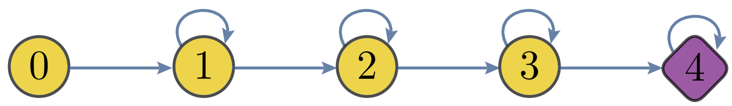
We're not entirely sure how they found so many places to go out to dinner. Perhaps they're just visiting Dairy Keen every night because it's a gym.

a. On the second night, what is the probability that the waiter will pick someone new that wasn't picked the night before?

b. Suppose that on the nth night, Amy, Brian, and Chris have already paid for dinner at least once. What is the probability that Danny's credit card will be chosen that night?

On the nth day of Christmas, my true love game to me: n recursive functions, $n-1$ recursive functions, $n-2$ recursive functions . . . two billy goats, and a hot tub closing at 10.

c. Here is a transition diagram for this situation. The number in each state indicates the number of people who have paid at least once. Label the arrows with the correct transition probabilities.



d. Use Problem 3 from Day 12 to determine the expected number of dinners until everyone has paid at least once.

$\frac{1}{p}$

3. Oh wait! There are actually two Amys, two Brians, two Chrises, and two Dannys at PCMI. Figure stuff out.

Bum ba BUUUMMMMM. (Cue the dramatic chipmunk.)

Your Stuff

Your jokes. **Our jokes.**

T1. Go back to the Opener on Day 12. Across all three games, count the number of zeros and ones, and enter the data into a spreadsheet. Compare your results to Problem 11 from that day. What do you notice? What connections can you make? What do you wonder?!

If you can read this, you're a 01101000 01110101 01100111 01100101 01101110 01100101 01110010 01100100.

T11. Let's play *Suzanne's Composite Mind Blower*!

- Roll a fair die.
- If a composite number comes up, you win. Otherwise, take another roll.

90% of all students think 1 is a prime number. 92% of all statistics are made up on the spot.

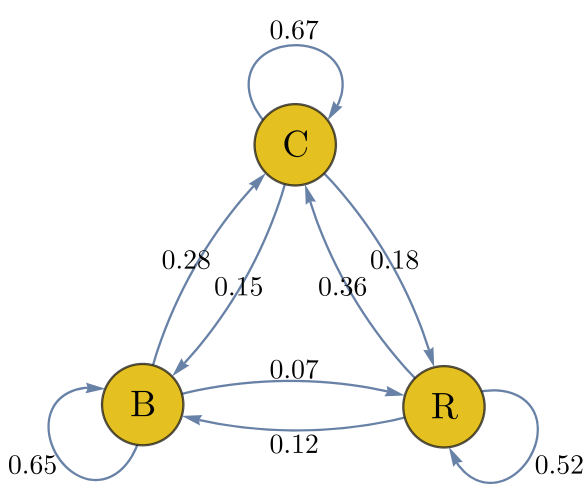
a. What is the expected number of rolls, R, until winning the game?

b. Create a transition diagram to model this situation. Label each arrow with its transition probability. Use this diagram to come up with an equation for R, the expected number of flips:

$$R = \square \cdot \square + \square \cdot \square$$

T12. In the rockchuck community here at Zermatt, there are three social classes: burrowers, chirpers, and royalty. The diagram below shows the probability of a rockchuck's offspring being in a social class, based on the class of their mother:

Replace "chirpers" with "peepers" and we could have a ton of PBR references. Maybe Table 12 just really likes Championship Bull Riding.



An initial study shows 21% of the population are burrowers; 68% are chirpers, and 11% are royalty.

This study was conducted via SurveyMonkey and has a ± 100 percent margin of error. Rockchucks with only a landline phone were not included.

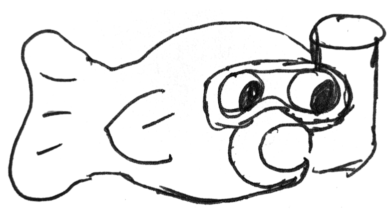
a. What will be the distribution in one generation?
b. What will be the distribution in two generations?
c. What will happen in the long run?
d. What would happen instead if the initial population was 100% burrowers?

T7. Set Theory and Awesome-Totes are working together in an escape room. To escape they must solve 3 puzzles in 60 minutes. They have to enter in their answers into a computer and they are allowed one entry every 2 minutes. Therefore, they have at most 30 chances to guess at the answers to the puzzles. Every time they guess, they have a 10% chance of getting a puzzle right. They have to start with the first puzzle and they can't move on to the subsequent puzzles until they've finished the puzzle they're working on. What is the probability that Set Theory and Awesome-Totes will escape?

Stop guessing! Also, the answer to the first puzzle isn't 39, 38, or 36. These teams would be much faster at puzzle solving if they didn't waste their summers playing volleyball.

T3. There are six types of fish in an endless bucket of fish: tuna, trout, halibut, cheddar Goldfish crackers, salmon, and bass. You are going to play fish roulette. You have an equal chance to pull each kind of fish. How many fish do you expect to pull until you have pulled a complete set of one of each type of fish?

Because it's all about that bass, about that bass, no sardines.



Back To Our Stuff

4. Cal solved Problem 1 in a slightly different way. He wrote

$$W = (6+1) \cdot \frac{1}{2} + (W+6+1) \cdot \frac{1}{2}.$$

Try to make sense of what he did.

Cal solved this problem by first driving an NCTM president around on a motorcycle. No sharks were jumped while performing this stunt, unfortunately.

5. For transition matrix T, consider

$$S = I + T + T^2 + T^3 + T^4 + \cdots$$

a. What is TS?
b. Solve for S. Careful, these are matrices! They might explode and kill us all.
c. For vector v, what is the value of

$$v + Tv + T^2v + T^3v + T^4v + \cdots$$

TS dares to eat a peach, but only if there isn't any bacon available for breakfast. Hopefully you can complete this pruf! It rocks.

6. Here's a particularly interesting T and v to consider:

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

a. What is $I - T$? What happens when you try calculating the inverse of $I - T$?
b. Prune matrix T so this doesn't happen anymore, creating matrix Q.
c. Armed with a new matrix, determine the sum of

$$v + Qv + Q^2v + Q^3v + Q^4v + \cdots$$

and interpret the results in the context of coin-flipping problems. Wow!

"What is $I - T$? It's $I - T$! What is $I-T$. It's $I-T$! What is $I - T$. You want $I - T$ all, but you can't have it. What an epic problem.

7. Use matrices and magic to show that the expected number of turns before HHT appears as a three-flip sequence is . . . hm, what is it? How many of these flips, on average, occur with no progress? With one step of progress? With two steps?

I just told you, it's it.

8. Solve some other three-flip sequences. Use the style of Problem 1, or Problem 6, or both! So cool!

9. Use matrices and magic to determine the expected length of the entire game when HHH plays HTT in *Beat Cal*. Compare to the work in Problem 17, a problem you should totally do if you didn't get to.

10. Use matrices and magic to determine the expected length of the volleyball game if Awesome-Totes leads

Flipping away, I know a sequence to play, I'll play it anyway: today's another day to beat Cal, trying to make . . . I'll be beating you this game, okay?

H H T (*H H T*)

H T H (*H T H*)

I'll be gone
In a day or twooooooo

4-3, must win by 2, when Set Theory has a 75% chance to win each point.

11. Use matrices and magic to determine the expected length of the credit card roulette game from Problem 2.

'Cos the roulette players play play play play play, and then Chris will probably pay pay pay pay pay, I'm just gonna . . . dangit, we got this far without any reference to Tom Hiddleston, we can make it to the end. Good road trip music though, I suppose.

12. **a.** If you haven't already done Problems 17 and 19 from Day 12, do them now.

b. How does using an unfair coin affect the expected length of the game? Is there a minimum or maximum expected length? How does the length of the game change as p changes?

13. There are 12 tables at PCMI. Imagine a large population at each table. When it's time to change tables, everyone does! They move to all neighbor tables—up/down/left/right— with equal probability. (Not all tables have the same number of neighbors.)

Don't forget B A Start Select. 30 lives!

Imagine everyone starts at Table 1. What happens in the long run? What if all tables started out equally populated?

14. The weather in Midway can be described by a 2×2 transition matrix with states "sunny" and "meh". At the steady state, the probability of any two consecutive days being sunny is 70%, and the probability of any two consecutive days being meh is 10%. What is the transition matrix?

As featured on the FXX show It's Always Sunny Or Meh In Midway.

15. Emily and Chris are playing rock-paper-scissors. A player winning a rock-paper-scissors game wins \$2 from their opponent.

In order to claim the prize, each player must tell each other, "I want my two dollars", then chase them down a ski hill. Fortunately we have lots of those here.

a. Think of an unbeatable strategy for rock-paper-scissors. By this we mean a strategy no other strategy can possibly outplay. The "unbeatable strategy" is not necessarily a strategy that will lead to victory.

b. The game has changed! Now, *anyone* playing rock must immediately pay \$1 to their opponent, in addition to any other wins and losses. Try to find an unbeatable strategy for this game.

16. Let T be the matrix from the Opener on Day 4. This matrix encodes information about how participants are redistributed among Tables A, B, and C during a PCMI table change.

a. Suppose all of the participants are initially at Table A. What percentage of participants will be at Table A, B, and C after two table changes?

b. Repeat part a assuming all of the participants are initially at Table B.

c. Repeat part a assuming all of the participants are initially at Table C.

d. Calculate the matrix T^2. Explain why this matrix is connected to the answers from the three previous questions above. How could T^2 be used to calculate what happens after two consecutive table changes?

Fun fact: Yoshi's full name is T. Yoshisaur Munchikoopas! That sounds very family-friendly.

17. Refer back to Problem T3. How many fish do you expect to pull until you have pulled *six fish of the same type*?

18. Use tactics like the one in Problem 10 from Day 12 to find an exact fraction equal to

$$\frac{1}{100} + \frac{1}{10000} + \frac{2}{10^6} + \frac{3}{10^8} + \frac{5}{10^{10}} + \cdots$$

What does the decimal expansion of this fraction look like?

Hint: it ends in "dot dot dot".

Totally Ridiculous Trig Stuff

19. Factor this:

$$z^{11} - 1 = (z-1)(z^5 + \text{stuff})(z^5 + \text{other stuff})$$

What does this have to do with Problem 21 on Day 7?

Ow my brain, make it stop. Oh! It stopped!

No More Stuff

20. Thanks. We had a wonderful time and hope you did too. See you again as soon as possible.

See you on the next exciting episode of *The Gone Show*! Oh wait, we're canceled.