Day 1: Over + Over Again

Welcome to PCMI! We know you'll learn a great deal of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about eigenvectors, think again: the problems should lead to the appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That's* why it's Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

PCMI participants have solved at least two previously unsolved problems presented in these courses.

When you get to Day 3, come back and read this again.

Consider this your first excursion into recursion . . .

Opener

1. We're going to start with doing the same thing, over and over. The *Fibonacci sequence* is one of the most famous sequences of all time. It starts with 0, then 1, then each new term is the sum of the two that come before it. A more formal definition is

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \quad \text{if } n > 1$$

For example, $F(2) = F(1) + F(0)$, then $F(3) = F(2) + F(1)$, then . . . Use the values of $F(0)$ and $F(1)$ to find $F(2)$, then use the values . . . then they tell two friends, and . . .

a. For the Fibonacci sequence, determine $F(0)$ through $F(9)$ and the sum of these ten numbers.

b. Your table will be given four new pairs of starting numbers. For each pair, determine the first ten numbers (including the two givens) and their sum. Notice anything?

c. Describe some similarities between the five sequences your table worked with.

Most of the time if we use $F(n)$ with capital F, we mean the "real" Fibonacci sequence, not these impostors.

Stuff in boxes is more important than other Important Stuff!

Important Stuff

2. Traci defines the sequence $0, 1, 2, 3, 4, \dots$ recursively:

$$t(0) = 0$$
$$t(n) = t(n-1) + 1 \quad \text{if } n > 0$$

a. For some number a, $t(a) = 23$. Find a.

b. Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(9)$.

c. Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(100)$.

3. Write a recursive definition for $a(n)$ that fits the sequence $2, 6, 10, 14, 18, \dots$

This means $a(0)$ should be 2, $a(1)$ should be 6, and $a(73)$ should be 294. Just sayin'.

4. Write a recursive definition for $b(n)$ that fits the sequence $2, 6, 18, 54, 162, \dots$

5. Without a calculator, *estimate* the number of digits in $F(100)$, a big Fibonacci number. Yes, it's fine to get this wrong! But think it over a bit.

Avoid saying "the 100th Fibonacci number" unless it's clear what you mean. $F(100)$ is usually called the 100th Fibonacci number, but it can be confusing.

6. Water you going to drink a lot of today?

7. Find two numbers with the given sum s and product p.

a. $s = 7, p = 10$	**e.** $s = 10, p = 23$
b. $s = 2, p = -3$	**f.** $s = 10, p = -1$
c. $s = -13, p = 30$	**g.** $s = 100, p = 2379$
d. $s = 10, p = 25$	**h.** $s = 100, p = 2017$

Neat Stuff

8. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

Some of the Fibonacci numbers can't even.

9. Which Fibonacci numbers are multiples of 3? Explain why this happens.

10. Naira's favorite sequence starts with $N(0) = 7$ and $N(1) = 4$. After that, each term is the opposite of the sum of the previous two terms. Write the first ten terms of this sequence.

We heard her call this the "neganacci" sequence.

11. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$L(2) = 3, L(3) = 4, L(4) = 7$. There's a lot of literature on Fibonacci and Lucas. We humbly request that you not read any of it for now, so that you have the chance to find and prove some of the results on your own.

$$\begin{aligned} L(0) &= 2 \\ L(1) &= 1 \\ L(n) &= L(n-1) + L(n-2) \quad \text{if } n > 1 \end{aligned}$$

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

12. Ramona's sequence is the sum of the Lucas and Fibonacci sequences.

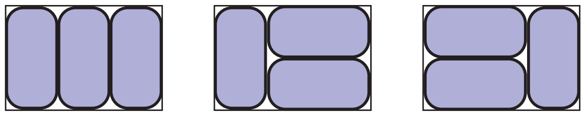
She likes this because $L + F = R$ when you're counting letters as money!

$$R(n) = L(n) + F(n)$$

Figure out what you can about Ramona's sequence, and any new relationships you can figure out between the Lucas and Fibonacci sequences.

13. Without a calculator, determine the units (ones) digit of $F(100)$.

14. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



15. Carla wrote this sequence for $c(n)$: $1, 2, 11, 43, 184, 767 \dots$ Find a recursive rule that could define Carla's sequence.

16. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

17. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting about it.

Well, duh, they have the common factor 1. (We mean "legitimate" common factors.)

Tough Stuff

18. Genevieve claims that starting with $F(7) = 13$, it's possible for $F(n)$ to be prime, but it's *never* possible for $F(n) + 1$ or $F(n) - 1$ to be prime. Prove this . . . well, if it's true . . .

19. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}}} = 15$$

20. Consider the unit circle $x^2 + y^2 = 1$. Plot n equally spaced points on the circle starting from $(1, 0)$. Now draw the $n - 1$ chords from $(1, 0)$ to the others. What is the product of the lengths of all these chords?

21. Take the diagram you drew in problem 20 and stretch it vertically so that the circle becomes the ellipse $5x^2 + y^2 = 5$. All the points for the chords scale too. What is the product of the lengths of all *these* chords?

What.

Day 2: Let The Good × Roll

Opener

1. We're going to start with doing the same thing, over and over. The *Monica sequence* is one of the least famous sequences of all time. It starts with 2, then 2 again, then each new term is *twice* the one before it, plus *three times* the one before that. A more formal definition is

$$M(0) = 2$$
$$M(1) = 2$$
$$M(n) = 2 \cdot M(n-1) + 3 \cdot M(n-2) \quad \text{if } n > 1$$

For example, $M(2) = 2 \cdot M(1) + 3 \cdot M(0)$, then $M(3) = 2 \cdot M(2) + 3 \cdot M(1)$, then . . . Use the values of $M(0)$ and $M(1)$ to find $M(2)$, then use the values . . . then lather, rinse, and . . .

a. For the Monica sequence, determine $M(0)$ through $M(8)$.

b. Your table will be given four new pairs of starting numbers. For each pair, determine the first nine numbers (including the two givens). Notice anything?

I noticed it was a vague question. I wonder if the questions will always be this vague.

c. Use your work to find a way to calculate $M(10)$ for the original Monica sequence directly without calculating $M(9)$.

d. Describe some similarities between the five sequences your table worked with.

Important Stuff

2. Find a solution to this system of equations:

$$A + B = 2$$
$$3A - B = 2$$

3. Vince gives you a new starting pair for today's Opener: $V = \begin{bmatrix} 0 \\ 40 \end{bmatrix}$. Determine the first nine numbers for this starting pair *super-quick*.

4. Kimberly adds starting pairs P and Q from today's Opener. What happens!

P is pretty normal!

5. Iris's sequence is $2, 5, 8, 11, 14, \ldots$.

This means $I(0)$ should be 2, $I(1)$ should be 5, and $I(221)$ is almost but not quite 666.

a. Describe the pattern of Iris's sequence:
*Each term in Iris's sequence is*____________________.

b. Write a recursive definition for Iris's sequence by filling in these boxes:

While "a number" is a valid answer . . . some answers are better than others. Sorry.

$$I(0) = \boxed{}$$
$$I(n) = I(n-1) + \boxed{} \quad \text{if } n > 0$$

6. Johnson's sequence is $3, 6, 12, 24, 48, \ldots$

a. Describe the pattern of Johnson's sequence:
*Each term in Johnson's sequence is*____________________.

While "more than the last one" is a valid answer . . .

b. Write a recursive definition for Johnson's sequence by filling in these boxes:

$$J(0) = \boxed{}$$
$$J(n) = \boxed{} \quad \text{if } n > 0$$

7. Calculate these.

a. $(5+\sqrt{2}) + (5-\sqrt{2})$
b. $(5+\sqrt{2}) \cdot (5-\sqrt{2})$

8. Find two numbers with the given sum s and product p.

Product P was a terrible cereal.

a. $s = 10, p = 25$
b. $s = 10, p = 24$
c. $s = 10, p = 23$
d. $s = 10, p = 22$
e. $s = 10, p = 21$
f. $s = 10, p = 20$
g. $s = 10, p = 1$
h. $s = 10, p = -1$
i. $s = 10, p = -299$
j. $s = 100, p = 2451$

Neat Stuff

9. Work through Set 1's Opener using the starting pair $F(0) = x$ and $F(1) = y$. What do you notice?

10. Here's a recursive definition for the sequence $0, 1, 3, 6, 10 \ldots$:

'Recursive" doesn't mean "cursive again."

$$s(0) = 0$$
$$s(n) = s(n-1) + n \quad \text{if } n > 0$$

a. Determine $s(9)$.
b. Determine $s(100)$.

11. A "Monica-like" sequence is defined by

$$\begin{aligned} R(0) &= 5 \\ R(1) &= 19 \\ R(n) &= 2R(n-1) + 3R(n-2) \quad \text{if } n > 1 \end{aligned}$$

Find a closed rule, such as $R(n) = 3^n$, that matches the sequence, then used the closed rule to find $R(10)$.

A *closed rule* is one like $R(n) = 3^n + (-1)^n$. It has no recursion, and it also has no recursion.

12. Find a solution to this system of equations:

$$\begin{aligned} A + B &= 5 \\ 3A - B &= 19 \end{aligned}$$

13. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

Fibonacci numbers are like marching soldiers in The Wizard of Oz. Oh, ee, oh . . . oh, ee, oh . . .

14. Which Fibonacci numbers are multiples of 5?

15. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$$\begin{aligned} L(0) &= 2 \\ L(1) &= 1 \\ L(n) &= L(n-1) + L(n-2) \quad \text{if } n > 1 \end{aligned}$$

$L(2) = 3, L(3) = 4, L(4) = 7$. The Lucas sequence lives on the second floor. Please don't read literature about Fibonacci and Lucas, so that you can find and prove results on your own.

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

16. Ramona's sequence is the sum of the Lucas and Fibonacci sequences.

$$R(n) = L(n) + F(n)$$

Ramona said something about starting pairs and indexes and doubling or something.

Figure out stuff about Ramona's sequence and relationships between the Lucas and Fibonacci sequences.

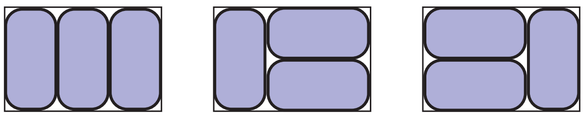
17. Here is the Zucanacci sequence. Figure stuff out!

$$\begin{aligned} Z(0) &= 2i \\ Z(1) &= 1 + i \\ Z(n) &= Z(n-1) + Z(n-2) \quad \text{if } n > 1 \end{aligned}$$

18. Write a recursive rule for $h(n)$ that fits the sequence $1, 10, 44, 160, 536, 1720, 5384 \ldots$

Roy recently learned the ending to Red Hot Chili Peppers songs can be recursive. Give it away now . . .

19. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



20. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

21. Without a calculator, determine the units (ones) digit of $F(100)$ and of $F(1000)$.

22. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

23. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting.

Don't count common factor 1. Well, maybe you can . . .

Tough Stuff

24. Describe a rule you could use to determine, given any integer $n > 1$, which Fibonacci numbers are divisible by n.

25. Prove that any positive integer can be written *in exactly one way* as the sum of one or more non-consecutive Fibonacci numbers. For example: $43 = 34 + 8 + 1$ while $43 = 21 + 13 + 5 + 3 + 1$ would be unacceptable.

There are 100 types of people in the world: those who understand the Zeckendorf representation and those who don't.

26. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}} = 1$$

Day 3: Mind ÷ Matter

Opener

1. We're going to start with doing the same thing, over and over. Here's a recursive definition for a function $J(n)$.

$$J(n) = 7 \cdot J(n-1) - 10 \cdot J(n-2)$$

Oyinka picks the starting data $O = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and calculates.

a. Determine $J(0)$ through $J(7)$ with Oyinka's starting data.

b. Oyinka says that her $J(n)$ grows exponentially. Is she right? Is she almost right?

c. Beth gives you four new sets of starting data. For each, determine the first eight numbers (including the two from the starting data). Notice anything?

d. Find a closed rule for Oyinka's $J(n)$, then use it to compute $J(8)$ directly.

This means that $J(0) = 2$ and $J(1) = 7$. To find $J(2)$, follow the recursion: $J(2)$ is 7 times $J(1)$, minus 10 times $J(0)$. The value of $J(2)$ is a very common age on dating profiles.

A *closed rule* is one like $J(n) = 3^n + (-1)^n$. It has no recursion, and it also has no recursion.

Important Stuff

2. Find two numbers with the given sum s and product p.

a. $s = 7, p = 10$
b. $s = 2, p = -3$
c. $s = 3, p = -10$
d. $s = 9, p = 14$
e. $s = 8, p = 15$
f. $s = 100, p = 2451$
g. $s = 200, p = 9991$
h. $s = 1, p = -1$

A similar problem was on Day 2's Important Stuff. If you haven't had a chance at that one, try it first, or don't, it's fine, I'm just a piece of paper.

3. The *common ratio* between two terms in a sequence is calculated by dividing a term by the one before it. Calculate the common ratios of Oyinka's $J(n)$, from $J(1)/J(0)$ up to $J(8)/J(7)$, to four decimal places. What up with that?

What up with that, I say, what up with that?!

4. This is Gareth's favorite recursive rule:

$$G(n) = 3G(n-1) + 10G(n-2)$$

Lynde picks the starting data $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

a. Determine $G(0)$ through $G(7)$ with Lynde's starting data.

b. Calculate the common ratio of consecutive terms. What's happening!!

Hey *hey* hey!

5. Angelina uses the recursion from problem 4 but she chooses starting data $\begin{bmatrix}1\\k\end{bmatrix}$. Oh noes! k is a variable!

a. Pick a number to use for k and determine $G(0)$ through $G(3)$. Do you think this function is exponential? Explain how you know.

b. Determine $G(0)$ through $G(3)$ in terms of k.

c. Find all possible values of k for which the starting data $\begin{bmatrix}1\\k\end{bmatrix}$ produces an exponential function.

If the Earth loses its protection from ultraviolet radiation, it will be the fault of the oh noes layer.

6. Find a closed rule for each of the four sets of starting data Beth gave you for $J(n)$ in the Opener.

7. There's a shorthand for the adding and scaling of starting data we've been doing:

$$\begin{bmatrix}1\\5\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\7\end{bmatrix} \quad \text{and} \quad 2\begin{bmatrix}2\\7\end{bmatrix} = \begin{bmatrix}4\\14\end{bmatrix}$$

Strawberry shorthand is the best flavor.

a. Find A and B so that $A\begin{bmatrix}1\\5\end{bmatrix} + B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\19\end{bmatrix}$.

b. A sequence starts with $\begin{bmatrix}5\\19\end{bmatrix}$ and follows the rule $J(n) = 7\,J(n-1) - 10\,J(n-2)$. Find a closed rule for this sequence.

Right now, hopefully you don't feel like it's 5:19 am.

c. Find A and B so that $A\begin{bmatrix}1\\5\end{bmatrix} + B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$.

d. Find A and B so that $A\begin{bmatrix}1\\5\end{bmatrix} + B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$.

e. Find A and B so that $A\begin{bmatrix}1\\5\end{bmatrix} + B\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}11\\-4\end{bmatrix}$. Use the last two!

Neat Stuff

8. There is a two-term recursive definition for $W(n)$ that fits the function $W(n) = 7^n - 2^n$. The rule is

$$W(n) = s \cdot W(n-1) + p \cdot W(n-2)$$

and s and p need to be found. To find s and p . . .

Two-terming could also be called two-timing, because it goes back in time twice. Just like Marty McFly!

a. Compute $W(0)$ through $W(4)$.

b. Here's a system of two equations

$$\begin{aligned} W(2) &= s \cdot W(1) + p \cdot W(0) \qquad \text{and} \\ W(3) &= s \cdot W(2) + p \cdot W(1) \end{aligned}$$

Solve the system to find s and p.

Psst: $W(0) = 0$ and $W(1) = 5$.

c. Verify that your recursive definition gives the correct values of $W(0)$ through $W(4)$.

9. **a.** Find a two-term recursive definition for $X(n)$ that fits the function $X(n) = 3^n + 5^n$.

b. Find a two-term recursive definition for $Y(n)$ that fits the function $Y(n) = 2 \cdot 3^n + 3 \cdot 5^n$.

c. Find a two-term recursive definition for $Z(n)$ that fits the function $Z(n) = 4 \cdot 3^n - 5^n$.

Two-terming is something presidents do, sometimes. But not always.

10. This is Hilda's favorite recursive rule:

$$H(n) = 5H(n-1) - 6H(n-2)$$

Find a closed rule for the sequence that fits each of these starting data.

a. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 \\ 13 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

11. What happens to the Fibonacci sequence if only units digits are considered? The sequence begins

$$0, 1, 1, 2, 3, 5, 8, 3, 1, \ldots$$

Some might say you are looking at things in "mod 10". Ignore those people and just do the problem.

12. Melvin wonders what happens when you take the product of two Fibonacci numbers that surround a third. What is it?

It's it. What is it? It's it.

13. In terms of n, how many ways are there to write n as the sum of ones and twos? Consider any reorderings to be *different* ways. There are three tilings, uh, ways to write 3 using ones and twos: $1+1+1$ or $1+2$ or $2+1$.

14. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

15. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 7 + \frac{-10}{r(n-1)} \quad \text{if } n > 0$$

Repeat for $r(0) = 2$. Neato.

16. Closed rules, closed rules!

Come get these *closed rules*! I know you want it (hey hey hey).

a. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ but this time for $t(n) = 5\,t(n-1) + 6\,t(n-2)$

b. $\begin{bmatrix} 2 \\ 100 \end{bmatrix}$ for $t(n) = 100\,t(n-1) - 2451\,t(n-2)$

c. $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ for $t(n) = 10\,t(n-1) - 23\,t(n-2)$

d. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $t(n) = \ t(n-1) + \ t(n-2)$

The Lucas sequence was once injured in a high school football game, but went on to star in "License to Drive."

Tough Stuff

17. Prove that the greatest common factor between $F(a)$ and $F(b)$ is also a Fibonacci number. But which one?

18. Find a two-term recurrence that has period 6: for any $n \geqslant 0$, $f(n+6) = f(n)$ and there is no smaller n for which this is true.

19. Find a two-term recurrence that has period 8.

20. What's this sum! $F(n)$ is the nth Fibonacci number.

$$\sum_{n=0}^{\infty} \frac{F(n)}{2^n} = \frac{0}{1} + \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots$$

Day 4: Genesis - Peter Gabriel = Genesis

Opener

1. We're going to start with doing the same thing, over and over. Find a closed rule for $T(n)$ satisfying

$$T(n) = 13\,T(n-1) - 30\,T(n-2)$$

with the starting data $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$. If you're not sure where to begin, calculate a bit, then try taking common ratios of consecutive terms.

This problem set is brought to you by Nelly, ft. Tim McGraw.

Important Stuff

2. Calculate each of these.

a. $\left(\dfrac{1+\sqrt{5}}{2}\right) + \left(\dfrac{1-\sqrt{5}}{2}\right)$

b. $\left(\dfrac{1+\sqrt{5}}{2}\right) \cdot \left(\dfrac{1-\sqrt{5}}{2}\right)$

Sheesh, fractions *and* radicals? It's only problem 2! And it's Sunday! And it's not even 9 am yet!

3. Find two numbers with sum $salt$ and product $pepa$.

a. $salt = 13, pepa = 30$
b. $salt = 10, pepa = 21$
c. $salt = 100, pepa = -1469$
d. $salt = 1, pepa = -1$

Whatta number, whatta number, whatta mighty good number!

If you get the last one right, you're golden. Problems about Spinderella may appear later.

4. The Hsu Shay Resort features 2 ski lifts, 1 cross-country trail, and 11 downhill ski runs connecting five locations: **B**ase of Ace, **R**aindeer Crossing, **I**cy Drink, **A**ltitude Sickness, and **N**ap Zone. There are ski runs with multiple difficulty levels between some of the locations.

Who say Hsu Shay open through May? Closed on Tuesday. Hsu Shay's loose hay today, you say? New day, who pay Hsu Shay? You? They? Touché.

The resort uses tokens as a form of payment: arriving at every location via skiing or lift requires one token and buying a coffee at the Base of Ace requires one token.

Cross-country skiing counts as skiing. Basically, following any arrow on the diagram costs one token.

a. Fill in this 5-by-5 grid showing the number of ways of traveling between any two locations, using exactly one token. There is a 1 in the upper-left corner

If you're taking the chair lift to the top, drink lots of water first!

because the only way to stay at B and spend exactly one token is to buy a cup of coffee.

Useless information *you should not use* when filling in this grid: Hsu or Shay might lie to you about their ski resort, but only when they are wearing red and not blue. Also, the coffee at Base of Ace will open up your eyes, but the shop can be a little hard to find; look for the sign.

... to location

# of ways from location

	B	**R**	**I**	**A**	**N**
B	1				
R					
I					
A	1	2	0	0	1
N					

... using exactly one token.

b. Julie spends 2 tokens to go from A to B. How many different ways could she have done this?

c. Natalie spends 2 tokens to go from B to A. How many different ways could she have done this?

d. Fill in this grid showing the number of ways of traveling between any two locations, using exactly two tokens. Complete the grid using only the grid for one-token travel. Do not use the map. Do not use any techmology.

Do not use a Phone-A-Friend, for that also requires techmology. Techmology is wack!

... to location

# of ways from location

	B	**R**	**I**	**A**	**N**
B				1	
R					
I					
A	9				
N					

... using exactly two tokens.

e. Build a grid showing the number of ways of traveling between any two locations, using exactly three tokens. Complete the grid using only the other grids.

Do not pass Go, do not collect $200, do not taunt Happy Fun Ball, do not mess with the Council of Ricks.

Neat Stuff

5. Here's a recursive rule:

$$S(n) = 10\,S(n-1) - 21\,S(n-2)$$

If Arundhati uses the starting data $\begin{bmatrix} 1 \\ k \end{bmatrix}$, what values of k will produce an exponential function?

6. Find the solution to each of these systems of equations.

a. $A\begin{bmatrix} 1 \\ 7 \end{bmatrix} + B\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

b. $A\begin{bmatrix} 1 \\ 7 \end{bmatrix} + B\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 19 \end{bmatrix}$

c. $A\begin{bmatrix} 1 \\ 7 \end{bmatrix} + B\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

d. $A\begin{bmatrix} 1 \\ 7 \end{bmatrix} + B\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$

Forgot what that stuff means? We defined this shorthand previously:

$\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and

$2\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$.

7. Here's a recursive rule:

$$T(n) = 10\,T(n-1) - 21\,T(n-2)$$

Hemang uses the starting data $\begin{bmatrix} h \\ k \end{bmatrix}$. Find a closed rule for $T(n)$ if . . .

a. $h = 2, k = 10$

b. $h = 1, k = 19$

c. $h = 0, k = 1$

d. you're not given h or k

The last answer will be in terms of h and k, which are grateful to finally be used for something other than horizontal and vertical shifts.

8. Oyinka's function $J(n)$ from yesterday satisfies

$$J(n) = 7\,J(n-1) - 10\,J(n-2)$$

with starting data $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

You saw that as n increases, the common ratio of consecutive terms approached 5. But what happens if n decreases?

a. Determine the value of $J(-1)$ that would allow the recursive rule to continue working. Specifically, $J(1) = 7\,J(0) - 10\,J(-1)$ gives this value.

b. Determine $J(-2)$ through $J(-6)$ to a reasonable number of decimal places.

c. What happens to the ratio of consecutive terms as n becomes more and more negative? The ratio is always more than 1, by the way.

Remember, the common ratio would be $J(-5)/J(-6)$, not the other way around.

9. Find closed rules for...

a. $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ for $R(n) = 10\,R(n-1) - 22\,R(n-2)$

b. $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ for $I(n) = 10\,I(n-1) - 29\,I(n-2)$

Deeper into the problem sets, things always seem to get more complex.

c. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $L(n) = L(n-1) + L(n-2)$

d. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ for $Y(n) = 6\,Y(n-1) - 9\,Y(n-2)$

Oh noes.

e. $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ for $Y(n) = 6\,Y(n-1) - 9\,Y(n-2)$

10. Research more sets of starting data or other recursions to figure out what's going on with $Y(n)$ in problem 9.

I say: hey-ey-ey-ey-ey, hey-ey-ey. I say hey! What's going on? And then I wake in the morning and I step outside, and I take a deep breath, and I get real high, and I scream from the top of my lungs: isn't this supposed to be the weekend?

Tough Stuff

11. What problem are we going to ask you to solve tomorrow? Fine, just solve it now, then.

12. The *Onemorenacci sequence* is defined by the rule

$$O(n) = O(n-1) + O(n-2) + 1$$

with starting data $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find a closed rule for the Onemorenacci sequence.

13. Prove that any positive integer can be written *in exactly one way* as the sum of one or more non-consecutive Fibonacci numbers. For example: $53 = 34 + 13 + 5 + 1$.

Day 5: (NA) ^ 16 = Batman

Opener

1. We're going to start with doing the same thing, over and over.

The *Lucas sequence* is defined by

Problem 2 from yesterday's set may be helpful.

$$\begin{aligned} L(0) &= 2 \\ L(1) &= 1 \\ L(n) &= L(n-1) + L(n-2) & \text{if } n > 1 \end{aligned}$$

Evaluate $L(0)$ through $L(6)$ and find a closed rule for $L(n)$.

2. We're going to start with doing the same thing, over and over. Wait... wait... What are these numbers called again? Fibboplonki? Nibbonoochie? Tribiani? Tamagotchi? Fonzarelli? Chimichanga? Minnelli?

Problem 2 from yesterday's set may be helpful.

$$\begin{aligned} F(0) &= 0 \\ F(1) &= 1 \\ F(n) &= F(n-1) + F(n-2) & \text{if } n > 1 \end{aligned}$$

Well, whatever they are, find a closed rule for them.

Important Stuff

3. Here are three pitchers' attributes of strength (STR), stamina (STM), skill (SKL), and speed (SPD).

Pitcher data provided by the Las Vegas 51s, whose slogan is "Always Bet On Grey".

	McGowan #19	Wheeler #47	Rowen #46
STR	3	4	20
STM	1	21	1
SKL	5	18	6
SPD	20	7	99

Elizabeth imagines a new pitcher, , a precise mix of these three: 20% McGowan, 35% Wheeler, and 45%

Hey Wheeler! You're a Lucas number *and* your digits are also Lucas numbers!

Rowen. Calculate the attributes of this new pitcher.

$$\text{STR} = \boxed{11} = .20 \cdot \boxed{3} + .35 \cdot \boxed{4} + .45 \cdot \boxed{}$$
$$\text{STM} = \boxed{8} = .20 \cdot \boxed{} + .35 \cdot \boxed{} + .45 \cdot \boxed{}$$
$$\text{SKL} = \boxed{} =$$
$$\text{SPD} = \boxed{} =$$

This new pitcher might answer to the name "Bzzz."

4. Are you missing your friends and family? PCMI sells care packages! They may contain bananas, raisins, ice cream bars, almonds, and napkins. There are five different kinds of care packages:

	Care Package				
	#1	#2	#3	#4	#5
Bananas	2	2	0	1	1
Raisins	3	0	0	3	0
Ice cream bars	5	2	1	2	1
Almonds	9	0	1	1	0
Napkins	4	0	0	3	1

Care packages may also contain traces of meat lasagna. Do not ask for your care package to be delivered "animal style".

Cynthia places the following order:

	# ordered
Care package #1	1
Care package #2	3
Care package #3	2
Care package #4	1
Care package #5	2

To fulfill Cynthia's order, how many of each item (bananas, raisins, etc.) need to be prepared?

Warning: ice cream bars may not be the smartest item to put in care packages. Fortunately, there are also napkins.

5. Tino and Cristina also order care packages:

	Tino's order	Cristina's order
Care package #1	0	10
Care package #2	0	30
Care package #3	0	20
Care package #4	2	10
Care package #5	0	20

To fulfill Tino's order, how many of each item (bananas, raisins, etc.) need to be prepared? Cristina's order?

6. Follow these steps to calculate this product of a matrix and a vector on an TI-Nspire CX:

A matrix is just an array of numbers. A vector is a matrix that has only one row or column.

There aren't enough TI-Nspires for everyone—please share or you can use some other technology that you're familiar with.

$$\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} .20 \\ .35 \\ .45 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

a. Press **on**. If not in calculator mode, press .

b. If someone had previously been in the midst of typing in a calculation to be performed, clear it by pressing **ctrl** then **del**.

c. Press to display a menu of templates. Use the directional arrows below the screen to highlight . Press **enter** to select it.

Please do not push ALT in between. Also, there is no ALT key on an Nspire.

d. A screen titled "Create a Matrix" will appear. Press **del** and **4** to change the number of rows to four. Press **tab** twice to highlight the "OK" button then press **enter** to select it.

When creating a matrix, be careful that it does not enslave humanity.

e. Fill in each of the 12 entries of the matrix by typing each one and pressing **tab** to move to the next entry. After the final entry, press **tab** to move your cursor to the right of the matrix. Do not press the multiplication key as it is not needed.

They say no one can be told what the matrix is . . . well, they're wrong, because clearly we're telling you right now.

f. Type in the vector by pressing and choosing again. This time, set the number of rows to 3 and the number of columns to 1. Fill in the numbers in that vector as before.

g. Press **enter** to calculate the product of the matrix and vector. Write the answer in the space above. What do you notice?

I noticed the instructions ended. I wonder how technology went from being wack yesterday to being good today.

7. Use technology (as you did in the previous problem) to check your work on problems 4 and 5.

8. Use technology to check your work on the Hsu Shay Resort problem from yesterday.

Sashay to Hsu Shay today, hoobae!

The Week In Review

9. Take a few minutes to look back at what you've done. List five things you learned this week, and two things you are still unsure about or would like to investigate further. We'll talk this over at the end of class.

$(NA)^8 + (HEY)^3 =$ GOODBYE

Neat Stuff

10. Here are the values of the first few numbers in the Lucas and Fibonacci sequences.

term	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$L(n)$	2	1	3	4	7	11	18	29	47	76	123	199	322	521	843
$F(n)$	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377

a. Find and describe the sums of corresponding Lucas and Fibonacci numbers.

b. Find and describe the products of corresponding Lucas and Fibonacci numbers.

c. Use the closed forms of Lucas and Fibonacci numbers to prove what you noticed about these products.

11. **a.** Something interesting happens when you take the product of two Fibonacci numbers that surround a third. What is it?

b. Something else interesting happens when you take the *sum* of two Fibonacci numbers that surround a third. What is it?

Whatizit was the horrible mascot for the Atlanta 1996 Olympics!

It's Crispin Glover's 2005 surrealist film, of course . . . which premiered in Park City!

12. The golden ratio ϕ is the number $\dfrac{1+\sqrt{5}}{2}$.

a. Show that $\phi^2 = \phi + 1$.

b. Show that $\phi^3 = \phi(\phi + 1)$ without evaluating ϕ.

c. Show that $\phi^3 = \text{blah}\phi + \text{bleh}$. You figure out the blahnks, but there's a catch: you are not allowed to write the symbol $\sqrt{5}$ anymore in this problem! Use the behavior of ϕ to guide you.

The correct pronunciation of ϕ^3 is "fum".

d. Show that $\phi^4 = \text{blih}\phi + \text{blöh}$, again without evaluating ϕ.

e. Show that $\phi^5 = \text{bluh}\phi + \text{blyh}$.

Hm, ϕ^4 can be broken down into smaller powers of ϕ . . .

f. Describe a general rule for ϕ^n. Awesome!!

g. Find cool rules for ϕ^n for *negative* values of n.

One starter is $\phi = 1 + 1/\phi$.

13. Let $f(n) = \phi^n + \phi^{-n}$. Use the results from problem 12 to evaluate $f(0)$ through $f(6)$.

$(\text{NA})^{11} + (\text{HEY})^1 = \text{JUDE}$

14. Calculate this expression to seven decimal places for $n = 8, 9, 10, 11$.

$$\frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$$

15. Problem 10 establishes that for Fibonacci numbers, $F(2n)$ is a multiple of $F(n)$. Find and prove a formula for this ratio:

$$\frac{F(3n)}{F(n)} =$$

16. **a.** How many ways can you pick a set of numbers from 1 to n with no consecutive numbers?

b. Solve it again with a new restriction: 1 and n are considered consecutive. Put another way: find the number of ways people could be sitting at a round table with n seats without anyone sitting next to anyone else, including the option that no one is sitting.

17. Find closed rules for...

a. $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ for $a(n) = 2\,a(n-1) - 1\,a(n-2)$

b. $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ for $a(n) = 2\,a(n-1) - 1\,a(n-2)$

c. What's goin' on? Can you prove it?

Brother brother brother, there's far too many of you dyin' . . .

18. Find closed rules for...

a. $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ for $b(n) = 10\,b(n-1) - 25b\,(n-2)$

b. $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ for $b(n) = 10\,b(n-1) - 25b\,(n-2)$

c. $\begin{bmatrix} 1 \\ 10 \end{bmatrix}$ for $b(n) = 10\,b(n-1) - 25b\,(n-2)$

d. What's going on? Research more starting pairs.

It's weird how both SuperHuman and The Gong Show can be good at the same time. I feel like if those shows were put together, an antimatter explosion would occur.

19. Let $f(n) = A\,f(n-1) + B\,f(n-2)$ be a two-term recurrence.

$(NA)^{10}$ = KATAMARI DAMACY

- **a.** Show that if $p(n)$ solves the recurrence, then so does $h \cdot p(n)$ for any constant h.
- **b.** Show that if $p(n)$ and $q(n)$ each solve the recurrence, then so does $h \cdot p(n) + k \cdot q(n)$ for any constants h and k.

20. Find a closed rule for $Q(n)$.

Things get messy in episodes where Q shows up.

$$\begin{aligned} Q(0) &= 8 \\ Q(1) &= 3 \\ Q(2) &= 79 \\ Q(n) &= 19\,Q(n-2) - 30\,Q(n-3) \qquad \text{if } n > 2 \end{aligned}$$

21. Algebraically prove each of these identities. What might they be useful for, pray tell?

- **a.** $x^n + y^n = (x+y)\left(x^{n-1} + y^{n-1}\right) - xy\left(x^{n-2} + y^{n-2}\right)$
- **b.** $Ax^n + By^n = (x+y)\left(Ax^{n-1} + By^{n-1}\right) - xy\left(Ax^{n-2} + By^{n-2}\right)$

Tough Stuff

22. Prove this amazing fact: $L(2n)$, $\frac{F(3n)}{F(n)}$, and $(L(n))^2$ are consecutive integers. It makes you wonder if there are more examples of this sort of thing . . .

23. Given any positive integer m, which Fibonacci numbers are multiples of m?

24. What happens to the Fibonacci sequence in mod m?

For example, in mod 7 the only numbers are 0 through 6. The sequence starts 0, 1, 1, 2, 3, 5, 1, 6, 0.

- **a.** Explain why it must be periodic and give a cap on this period in terms of m.
- **b.** Find the period of the Fibonacci sequence for various m, looking for any patterns and conjectures.

25. Generalize problem 15 to the ratio $\dfrac{F(mn)}{F(n)}$ for any positive integer m.

Ask Gabe why you should be watching "The Genius"! It's really good!

26. Evaluate this sum:

$$\frac{F_0}{1} + \frac{F_1}{10^3} + \frac{F_2}{10^6} + \cdots + \frac{F_n}{10^{3n}} + \cdots$$

Day 6: Fire Burning on the ⌊Dance⌋

Opener

1. The *rhyme scheme* of a poem describes which lines of the poem rhyme with each other. For example, limericks are five-line poems with rhyme scheme AABBA. Example:

A dozen, a gross, and a score,
plus three times the square root of four,
divided by seven,
plus five times eleven,
is nine squared and not a bit more.

The 1st, 2nd and 5th lines rhyme and the 3rd and 4th lines rhyme, but the 1st and 3rd lines don't rhyme.

- Each line of the poem is assigned a letter so that all lines with the same letter rhyme with one another.
- The first time that a letter is used in a rhyme scheme, it must be the earliest letter in the alphabet yet to be used.

a. List four possible rhyme schemes for five-line poems, and four *invalid* rhyme schemes for five-line poems.

b. Determine the total number of possible rhyme schemes for five-line poems.

There once was a lovely young fellow
who loved to eat roasted marshmallow
When he turned from the fire
He felt flames getting higher
Now his buns are all toasty and yellow.

ABCDE is a valid rhyme scheme. Whether it's a good poem is a different question.

Think about how to organize your work so that you can be sure that you've found them all.

Important Stuff

2. Multiply these. Use technology such as the Nspire (see problem 6 on Day 5) whenever you want, but look for ways to save time and energy.

Save time by traveling at relativistic speeds! Save energy by traveling at zero speed.

a. $\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

b. $\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} 40 \\ 20 \\ 40 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$

c. $$\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} .4 & 40 \\ .2 & 20 \\ .4 & 40 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ & \\ & \\ & \end{bmatrix}$$

d. $$\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

Save time by using a blue box that is smaller on the outside. Save energy by adding more mass!

e. $$\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

f. $$\begin{bmatrix} 3 & 4 & 20 \\ 1 & 21 & 1 \\ 5 & 18 & 6 \\ 20 & 7 & 99 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \\ & & \end{bmatrix}$$

3. Siaka knows this rule that takes points and produces new ones in the plane:

I think they met in college, or something.

$$(x, y) \mapsto (y, 6x + y)$$

a. Make a simple shape in the coordinate plane. Draw the shape that results after applying the rule. Is the new shape larger or smaller than the original?

b. A *fixed point* is a point for which (a, b) maps to itself under the rule. Determine all fixed points, if any, for this rule.

c. A *scaled point* is a point for which (a, b) maps to (ka, kb) under the rule, where k is a number. Show that $(1, 3)$ is a scaled point for this transformation.

Fixed points are scaled points, because they work when $k = 1$.

d. Find and graph all scaled points for this rule.

4. **a.** How many rhyme schemes are possible for a five-line poem whose scheme starts with AABB? ABAC?

b. Complete this table of the number of possible rhyme schemes:

It must be a table for ants! How could it possibly fit on this page!

	# of letters used					
	1	2	3	4	5	6
1-line poems	1					
2-line poems	1	1				
3-line poems	1		1			
4-line poems						
5-line poems		15				
6-line poems						

I still say only ants could fit in that table.

c. Look for patterns in the table that can be justified using what you know about rhyme schemes.

This is not the time for closed forms, if that's what you're thinking.

Neat Stuff

5. Melissa wants to write a seven-line poem with a new rhyme scheme each day. How long can she last without repeating any rhyme scheme?

6. Jace knows another rule that takes points and produces new ones: $(x, y) \mapsto (y, x + y)$. Triangle FLO has points $F(1, 1), L(2, 1), O(0, 0)$.

They met on a disastrous Tinder date, but stayed friends.

a. Graph triangle FLO. Graph the shape that results after applying the rule. Is the new shape larger or smaller than the original?

Are you feeling the FLO? Or saving money on car insurance? Or rounding down?

b. Graph the shape that results after applying the rule a second time.

c. . . . a third time . . . a fourth time.

d. What's happening? Any thoughts about why this is happening?

7. Two days ago Table 11 suggested finding a three-term recurrence relation

The T stands for Table 11. T also stands for Terry Tao, and for Tschebysheff!

$$T(n) = A\,T(n-1) + B\,T(n-2) + C\,T(n-3)$$

satisfied by the sequence $T(n) = 2^n + 3^n + 5^n$. Try it!

8. It's a terrible day at Hsu Shay: only Altitude Sickness and the Base of Ace are open. Any path that involves the other locations is out of order.

Mark has ten tokens to spend. How many different ways can he spend all ten tokens?

To clarify, "coffee coffee up down" and "up down coffee coffee" are different.

9. Start with the point $(-8, 5)$ and follow the recursion $(x, y) \mapsto (y, x+y)$ for a while. Plot all the points you find in this way. Describe the path taken by these points, and the path taken by points that come *before* $(-8, 5)$ under the same recursion.

What point (x, y) maps to $(-8, 5)$?

10. Find a closed rule for $Q(n)$ given

$$Q(n) = 19\,Q(n-2) - 30\,Q(n-3)$$

and the starting data $Q(0) = 8, Q(1) = 3, Q(2) = 79$.

Huh. Feels like there should be a $Q(n-1)$ poking around here somewhere.

11. Say, here's an interesting rule:

$$Q(n) = 3\,Q(n-1) - 3\,Q(n-2) + Q(n-3)$$

a. Use the starting data $Q(0) = 0, Q(1) = 1, Q(2) = 4$. What happens?

b. Try a different set of starting data and graph $Q(n)$. What do you notice?

c. . . . what in the heck is going on here?

Nothing happens, not even if you type XYZZY. You are in a maze of twisty little problems, all different.

Tough Stuff

12. The *Twomorenacci sequence* is defined by the rule

$$T(n) = \begin{cases} 0, & n = 0 \\ 1, & n = 1 \\ T(n-1) + T(n-2) + 2, & n > 1 \end{cases}$$

Find a closed rule for the Twomorenacci sequence.

Is that a twomore? *It's not a twomore!!* (Best said in an Austrian accent.)

13. Experiment with the recurrence relation

$$C(n) = 2x \cdot C(n-1) - C(n-2)$$

and the starting data $C(0) = 1, C(1) = x$.

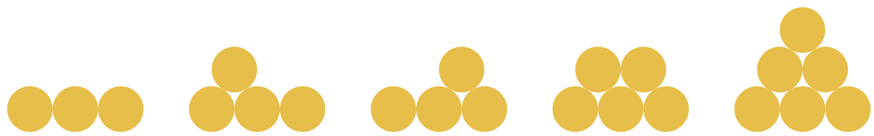
The C stands for craziness. C also stands for Chebyshev!

Day 7: ⌈Dancing⌉

Opener

1. Danny lives in two-dimensional Flatland. He's stacking coins on top of each other. Here are all five ways he can stack coins on top of each other, with 3 coins on the bottom row.

In Flatland, apartments are called "lorries".



   a. How many different ways can Danny create stacks of coins if he starts with a row of four coins on the bottom?

   b. Complete this table giving the number of ways Danny can create stacks of coins, based on the number of coins in the bottom row.

We can't say enough about how shockingly difficult it is for Danny to build these structures, living in two dimensions. He can't even see most of them when they're finished. So be careful when you describe his work as "plane".

# of coins in bottom row	# of possible coin stacks
1	
2	
3	5
4	
5	

Important Stuff

2. **a.** Using the one-token grid from problem 4a on Day 4, explain why there are exactly 5 ways to go from Icy Drink to Base of Ace using exactly two tokens.

   b. Think of the grids from the Hsu Shay Resort problem (Day 4) as 5-by-5 matrices. What happens when you square the matrix represented by the one-token grid? What happens when you cube the matrix represented by the one-token grid? Verify your conjectures (with or without technology).

The operators of Hsu Shay definitely have a profit motive! They say that all that Shay wants is another token, he's gone tomorrow, boy.

3. Here are the matrices from Day 4 showing the number of ways a person could travel between locations of the Hsu Shay Resort using one, two, or three tokens.

How can we be expected to follow these matrices . . . if they can't even fit inside the page??

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 11 & 2 & 1 & 2 & 1 \\ 6 & 6 & 0 & 3 & 3 \\ 17 & 4 & 1 & 6 & 3 \\ 12 & 2 & 0 & 10 & 2 \\ 9 & 6 & 1 & 4 & 3 \end{bmatrix}$$

two-token grid one-token grid three-token grid

The second column of the three-token grid is double the fourth column of the two-token grid.

a. Using the resort map, explain why this happens.
b. Using the matrices, explain why this happens.
c. Look for some other relationships of this kind.

The second column of the matrix represents Raindeer Crossing. Or maybe Raisins.

4. **a.** Describe the result of multiplying this matrix and vector.

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The result is tangy, with an oaky finish. Also the result is a vector.

b. Describe the result of multiplying these matrices.

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This time the result is a combination of many factors, most of them linear, with a silky smooth rectangular exterior associated with blue and red pills.

5. **a.** Draw a triangle with points $S(0.2, 2), A(1, 2), M(3, 6)$. Determine the area of the triangle.

b. Transform *SAM* according to the rule

$$(x, y) \mapsto (y, -10x + 7y)$$

Find the coordinates of the three new points.

c. Draw the new triangle and compute its area. How does the new area compare to the original?

d. Compute this multiplication. What do you notice?

$$\begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} 0.2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$$

You'll have to decide whether Sam is an Autobot or a Decepticon or perhaps just a LaBeouf. As long as he makes the transforming noise, does it really matter?

Neat Stuff

6. Crhyme schemes are like regular rhyme schemes, with one additional rule.

- Each line of the poem is assigned a letter so that all lines with the same letter rhyme with one another.
- The first time that a letter is used in a Crhyme scheme, it must be the earliest letter in the alphabet yet to be used.
- *No letter can be more than one higher (later in the alphabet) than the letter immediately before it.*

just like in high school
not everything rhymes
including this poem
with crhyme scheme ABCD

Every Crhyme scheme is also a valid rhyme scheme, but not necessarily vice versa. For example, ABCAC is a rhyme scheme, but it isn't a valid Crhyme scheme.

a. How many four-line Crhyme schemes are there?

b. Complete this chart of the number of valid Crhyme schemes.

all i know is
i would have paid more
attention
in english class
if there were more stuff
about crime schemes

	Last letter in the scheme						
	A	B	C	D	E	F	total
1-line poems	1						
2-line poems	1	1					
3-line poems	2		1				5
4-line poems							
5-line poems	14						
6-line poems							

7. Marissa thinks there may be a way to use resort-like maps and matrices to figure out the number of Crhyme schemes. What do you think? Let's try it!

8. Linda wants to write a seven-line poem with a new Crhyme scheme each day. How long can she last without repeating?

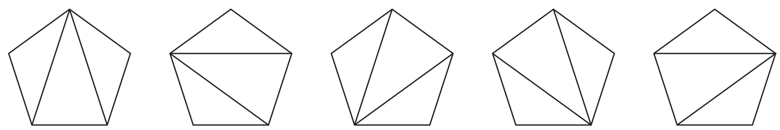
today my father bought a
dvd of law and ordah . . .

9. Establish a one-to-one correspondence between each of Danny's coin stacks (with a bottom row of n coins) and a Crhyme scheme for a n-line poem.

Pinball tip of the day: tilting Danny's coin stacks 30 degrees clockwise may offer some insight, but will also end your ball and you will receive no bonus.

10. Here are all the ways that a regular pentagon can be cut into 3 triangles by drawing two non-intersecting diagonals.

These ways recently became the hosts of some show on Fox News.



Count the number of ways that a regular $(n+2)$-gon can be cut into n triangles by drawing $(n-1)$ non-intersecting diagonals. Hint: Once you get to the regular hexagon, don't forget to include the triangulations that create an equilateral triangle within the hexagon.

11. The recursive rule $(x, y) \mapsto (y, -21x + 10y)$ takes a point and produces a new point. Repeating this recursion gives a sequence of points, as long as you have starting data For each set of starting data, find the next three points in the sequence.

Beware, using the point $(4, 20)$ may result in classroom snickering.

a. $(2, 10)$
b. $(4, 20)$
c. $(\frac{10}{21}, 2)$
d. $(1, 3)$
e. $(a, 3a)$
f. $(1, 7)$

12. **a.** Build some other polygons and transform them according to the rule in problem 5. What happens to the shape of the polygons? What happens to the area of the polygons?
b. Find all the *scaled points* for this recursion, points (a, b) taken to (ka, kb) for some number k.

The sound effect is a little like "wuh-wuh-wah-whah" but there are many variations.

13. In a circle with diameter 100, a chord of length 2 is drawn. Point P is the midpoint of the chord. What is the shortest distance from P to the circle's circumference? Give your answer as a decimal to 26 places. Wait, *what*?

Really, 26 decimal places?? Why would you want me to . . . oh my gravy what is going on here

Tough Stuff

14. Explain why that just happened. (The thing in problem 13.)

Day 8: [You Spin Me Right]

Opener

1. Put three White balls and three Grey balls into a cup. You'll pull the balls randomly out of the cup one by one, keeping score with the number of White and Grey balls you've pulled. Grey wins the game by taking the lead at any time. White wins only if they do not trail *at all* while all six balls are pulled.

For three minutes, run this game as many times as you can, tallying the number of wins for White and Grey.

It's a battle! White is good at chemistry, while Grey is good with ropes.

If the first ball is Grey, stop! Grey just won 1-0.

3

# of games in which White wins	# of games in which Grey wins	total # of games

2. This time, put four White balls and four Grey balls in your cup. The game rules are the same as before.

a. Before you begin, make a prediction about whether White is *more* or *less* likely to win with 4 balls, compared to 3.

b. For three minutes, run this game as many times as you can, tallying the number of wins for White and Grey.

White said something about being the one who knocks. Grey misinterpreted this as something altogether different.

Grey is Canadian, and said that the winner of this game should receive some sort of Grey Cup. That's pretty presumptive of a win . . .

4

# of games in which White wins	# of games in which Grey wins	total # of games

3. Repeat for five White balls and five Grey balls.

5

# of games in which White wins	# of games in which Grey wins	total # of games

Important Stuff

4. When your table has finished all three games, combine your data and pick someone to enter it here:

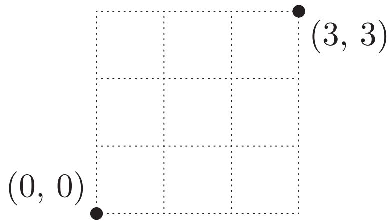
`http://bit.ly/pcmi2017`

Estimate the overall probability that White wins each game.

White says we'd better call Saul for this, while Grey put someone named Anastasia in charge of the business side of things.

5. Brooke wants to get from the origin to $(3,3)$ in the plane. She can only take single steps, one space to the right or one space up.

Different paths can intersect: "up up right right" is different from "up right up right". Sadly for Konami players, down and left are not allowed.



a. How many different ways can Brooke travel from $(0,0)$ to $(3,3)$?

b. How many different ways can Brooke travel from $(0,0)$ to $(4,4)$?

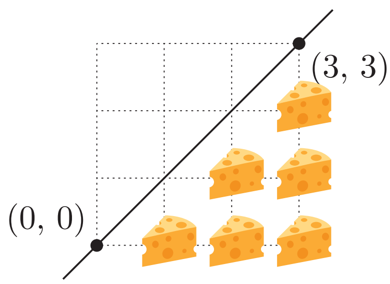
c. . . . from $(0,0)$ to $(5,5)$?

When Brooke travels to $(4,4)$, she doesn't have to go through $(3,3)$.

6. Kate also wants to get from the origin to $(3,3)$ in the plane. She takes single steps, one space to the right or one space up. Unlike Brooke, Kate is deathly afraid of cheese, which occupies all of the grid points strictly below the diagonal line connecting the origin to $(3,3)$.

There was an incident involving a cheesehead. Let's not ask.

She can be *on* the diagonal line, she just can't be *below* the diagonal line.



a. How many different ways can Kate travel from $(0,0)$ to $(3,3)$ that avoid the cheese?

b. . . . from $(0,0)$ to $(4,4)$?

c. . . . from $(0,0)$ to $(5,5)$?

What cheese is made backwards? . . . Edam. Man that pun was really . . . pretty awful.

7. Brooke has a list of all the ways she could travel from $(0,0)$ to $(3,3)$. Kate picks one of these ways at random.

a. What is the probability that Kate picks a route to $(3,3)$ that avoids the cheese? Use your previous work to help!

b. What changes if Brooke's routes go to $(4,4)$?

c. . . . to $(5,5)$?

If Kate picks successfully, the cheese will stand alone. Hi ho, the derry-o. And yes, that really is how to spell that line.

Neat Stuff

8. Here's a rule that maps points into new ones:

$$(x, y) \mapsto (y, -10x + 7y)$$

a. Find a point $(1, k)$ that gets doubled to $(2, 2k)$ by this rule.

b. Find some points (x, y) that get doubled by this rule. In other words, find (x, y) that map to $(2x, 2y)$.

c. Find some points (x, y) that get tripled by this rule.

d. Find some points (x, y) that get quintupled by this rule.

But but you said (x, y) maps to $(y, -10x + 7y)$. You're saying it could map to *both* that and $(2x, 2y)$ at the same . . . ohhh.

9. Here are the matrices from Day 4 showing the number of ways a person could travel between locations of the Hsu Shay Resort using one, two, or three tokens.

$$\underset{\text{two-token grid}}{\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix}} \quad \underset{\text{one-token grid}}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{three-token grid}}{\begin{bmatrix} 11 & 2 & 1 & 2 & 1 \\ 6 & 6 & 0 & 3 & 3 \\ 17 & 4 & 1 & 6 & 3 \\ 12 & 2 & 0 & 10 & 2 \\ 9 & 6 & 1 & 4 & 3 \end{bmatrix}}$$

The fourth column of the three-token grid is the sum of the first and third columns of the two-token grid.

a. Using the resort map, explain why this happens.

b. Using the matrices, explain why this happens.

c. Look for some other relationships of this kind.

Hsu Shay is branching out into whip and boot manufacturing, and services for teenage love and stalking. Regardless, it's better than free fallin'.

10. a. List all the possible ways White can win the three-ball game from the Opener.

b. List all the possible ways White can win the four-ball game from the Opener.

c. Connect this work to the work from Day 7's coins and/or Crhyme schemes.

Assume White uses only legit, legal means to win, and that none of the balls have been crushed into a crystal form.

11. How many of Kate's paths to $(5, 5)$ go through $(4, 4)$? How many go through $(3, 3)$? Look for some possible ideas here to help describe a recursive rule for the total number of these paths.

Breaking up the problem into smaller ones, hmm. Interesting.

12. Here are the matrices from Day 4 showing the number of ways a person could travel between locations of the Hsu Shay Resort using one, two, or three tokens.

Weekend trips to Hsu Shay are finally available for PCMI participants! Sign up on the board. Please no more transcendental or imaginary numbers of participants, unless your car is specifically built to handle these numbers.

$$\underset{\text{one-token grid}}{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix}} \quad \underset{\text{two-token grid}}{\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix}} \quad \underset{\text{three-token grid}}{\begin{bmatrix} 11 & 2 & 1 & 2 & 1 \\ 6 & 6 & 0 & 3 & 3 \\ 17 & 4 & 1 & 6 & 3 \\ 12 & 2 & 0 & 10 & 2 \\ 9 & 6 & 1 & 4 & 3 \end{bmatrix}}$$

The second row of the three-token grid is triple the first row of the two-token grid.

- **a.** Using the resort map, explain why this happens.
- **b.** Using the matrices, explain why this happens.
- **c.** Look for some other relationships of this kind.

13. For each row of Pascal's Triangle, calculate its mean squared value.

For example, the *values* in one row are 1, 4, 6, 4, 1. The *mean squared value* is the mean of . . . something.

14. For each recursion on points, find all the *scaled points*, points where (x, y) maps to a multiple of itself (kx, ky).

- **a.** $(x, y) \mapsto (2x, 2y)$
- **b.** $(x, y) \mapsto (x, 2y)$
- **c.** $(x, y) \mapsto (-x, y)$
- **d.** $(x, y) \mapsto (y, -21x+10y)$
- **e.** $(x, y) \mapsto (y, 3x + 2y)$
- **f.** $(x, y) \mapsto (y, x + y)$

Tough Stuff

15. Show how to divide Brooke's paths from $(0, 0)$ to (n, n) into $(n + 1)$ equal sets, exactly one of which is the set of Kate's paths.

16. Daniel, Daniel and Daniel give you a fun nonlinear recurrence:

Sure the recurrence is fun, but which Daniels are the fun Daniels? Can't they all be fun? No, they cannot.

$$\varepsilon(n) = \varepsilon(n-1) \cdot (2 - \ln \varepsilon(n-1)) \quad \text{with} \quad \varepsilon(0) = 1$$

What is $\varepsilon(1)$? What is $\varepsilon(5)$? What's going on?

17. Find a rule like the ones in problem 14 so that every point (x, y) maps to itself after exactly 5 iterations.

And no, $(x, y) \mapsto (x, y)$ is not allowed. This is Tough Stuff, fool.

Day 9: What Is The Point??

Opener

1. We're going to start with doing the same thing, over and over. Here's a rule that maps points into new ones:

Today's problem set is sponsored by Nathan Sykes. Also,

$$(x, y) \mapsto (4x + 2y, x + 3y)$$

a. Triangle EAT has points $E(0, 1)$, $A(-1, 1)$, $T(-1, 3)$. Determine the area of the triangle.

b. Transform triangle EAT according to the rule. What are the new coordinates and new area?

EAT again? We just had breakfast.

c. Transform a second time. What are the new coordinates? What's your guess about the new area?

d. What points (x, y) get scaled by a factor of 2 by this rule? There is more than one!

Such points would get mapped from (x, y) to $(2x, 2y)$. But we also said the rule maps (x, y) to $(4x + 2y, x + 3y)$. Oh!!

e. What points (x, y) get scaled by a factor of 3 by this rule? There isn't more than one!

f. What points (x, y) besides the origin get scaled by a factor of k by this rule? Find both k and the points that go along with those values of k. Within your table, have some people try to determine the values of k first, while others try to determine the points first.

Yes, you are trying to "solve" two equations involving three variables. Don't worry—it will all work out.

Important Stuff

2. **a.** The unit square has corners $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. What's the area of the unit square?

b. Transform the unit square using the rule above. What shape is formed and what is its area?

3. This is your last chance. After this, there is no turning back. You take the blue pill: the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill: you stay in Wonderland, and I show you how deep the rabbit hole goes. Remember: all I'm offering is the truth. Nothing more.

Which pill will you EAT? Fortunately, the pills aren't triangular.

Day 9: What Is The Matrix??

Opener

1. We're going to start with doing the same thing, over and over. Here's a rule that maps points into new ones:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Today's problem set is sponsored by forks. There is no spoon, but there are forks!

a. Triangle ATE has points $A \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $T \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $E \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Determine the area of the triangle.

Good thing you ATE that red pill.

b. Transform triangle ATE according to the rule. What are the new coordinates and new area?

c. Transform a second time. What are the new coordinates? What's your guess about the new area?

d. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ get scaled by a factor of 2 by this rule? There is more than one!

Such points would get mapped from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. But we also said the rule maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 4x+2y \\ x+3y \end{bmatrix}$. Oh!!

e. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ get scaled by a factor of 3 by this rule? There isn't more than one!

f. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ besides $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ get scaled by a factor of k by this rule? Find both k and the points that go along with those values of k. Within your table, have some people try to determine the values of k first, while others try to determine the points first.

Yes, you are trying to "solve" two equations involving three variables. Don't worry—it will all work out.

Important Stuff

2. a. The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's the area of the unit square?

b. Transform the unit square using the rule above. What shape is formed and what is its area?

3. **a.** If $v(0) = 7$ and $v(n) = 5v(n-1)$, find a closed rule for $v(n)$.

b. If $v(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $v(n) = 5v(n-1)$, find a closed rule for $v(n)$ that looks like $v(n) = \boxed{}\, v(0)$.

c. If $v(0)$ is any vector and $v(n) = Av(n-1)$, find a closed rule for $v(n)$ that looks like $v(n) = \boxed{}\, v(0)$.

Phew, I was expecting another blank page then questions about triangle TEA.

A could be a number or it could be a matrix! But it can't be a matrice, because that doesn't exist.

4. Here's a rule that maps points into new ones:

$$v(n) = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} v(n-1)$$

Here, $v(n)$ is a vertical vector like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a. Transform the unit square using the rule above. What shape is formed and what is its area?

Whoa.

b. Starting with $v(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Find a vector to use for $v(0)$ so that $v(1)$ is a multiple of $v(0)$. Then compute $v(2)$ and $v(3)$ for that vector.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

Use a method that is different from the one that you used in problem 1f.

Is the matrix real? If "real" is what you can feel, smell, taste and see, then "real" is simply electrical signals interpreted by your brain. Your mind makes it real. No one can be told what the matrix is. You have to see it for yourself.

5. Here's another rule that maps points into new ones:

$$v(n) = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix} v(n-1)$$

Here, $v(n)$ is a vertical vector like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a. Transform the unit square using the rule above. What shape is formed and what is its area?

Woah.

b. Starting with $v(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Find a vector to use for $v(0)$ so that $v(1)$ is a multiple of $v(0)$. Then compute $v(2)$ and $v(3)$ for that vector.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

e. Compute this matrix product for $n = 1, 2, 3$. What do you notice?

$$\begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}^n \begin{bmatrix} 0 & 1 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

Pretty sure the only Matrix product is batteries, right? Something like that. Hey, that movie is old enough to vote now.

Neat Stuff

6. Give an example of two matrices A and B such that the product AB can be computed but BA cannot.

Finally, proof that the Matrix is not all-powerful.

7. Ayesha multiplies two matrices and gets this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Oops, the middle matrix is missing! What is it!

8. This mathematical expression is invalid even though it has two opening and two closing parentheses:

$$(4+5))1-2($$

Ignoring the mathematical stuff apart from the parentheses, there are only two ways that a valid mathematical expression can use two sets of parentheses:

$$()() \qquad \text{or} \qquad (())$$

That's right! Ignore the mathematical stuff! Free your mind!

a. How many valid ways can three sets of parentheses be used in a mathematical expression?

b. . . . four?

c. Based on your work over the last few days, make a conjecture for the number of valid ways to that five sets of parentheses can be used, and look for an explanation of why this connection can be made.

Conjectures that the world is a machine-created illusion will be resolved with kung-fu fighting, I guess.

9. **a.** David has a vector. After using the map in problem 4, he got $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$. What's his vector?

b. Jennifer has a vector. After using the map in problem 4, she got $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What's her vector?

I'm trying to free your mind. But I can only show you the door. You're the one that has to walk through it.

c. Jasper has a vector. After using the map in problem 4, he got $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's his vector?

d. Diana has a vector with variables! After using the map in problem 4, she got $\begin{bmatrix} a \\ b \end{bmatrix}$. What's her vector?

Really, Diana? Your vector has variables?? What.

10. **a.** Find a 2-by-2 matrix A that takes any point in the plane and rotates it 90 degrees clockwise.

b. Compute A^2 and A^4.

c. Find a 2-by-2 matrix B that takes any point in the plane and rotates it exactly 60 degrees clockwise.

d. Find a 2-by-2 matrix C . . . 45 degrees clockwise.

Do not attempt to "find" the matrix. The Matrix is everywhere.

11. **a.** After expanding $a(a+b)(a+b+c)$ and combining like terms, how many different terms result?

b. Repeat for $a(a+b)(a+b+c)(a+b+c+d)$. Hmmm.

Please do not just write this bigger.

12. Did you know the numbers from problem 8 are lurking in even rows of Pascal's Triangle? Connect what you find to this expression from Day 8:

$$\frac{1}{n+1}\binom{2n}{n}$$

You can make a difference to find them!

Tough Stuff

13. **a.** Find a 2-by-2 matrix D that takes any point in the plane and rotates it exactly 15 degrees clockwise.

b. Find a 2-by-2 matrix E that takes any point in the plane and rotates it exactly 7.5 degrees clockwise.

14. For nonnegative integers n, calculate

$$\frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\,dx$$

Day 10: The Matrix, Reinputted

Opener

1. Here's a recursive rule: $v(n) = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} v(n-1)$

Hopefully you won't react to this problem set the way that most people reacted to the second Matrix movie.

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

b. Starting with $v(0) = \begin{bmatrix} 20 \\ 40 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Starting with $v(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ compute $v(1), v(2), v(3)$. Hmmmm.

It's just another of those things that make you go hmmmm.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

e. Compute this matrix product for $n = 1, 2, 3$. Hmmmmm.

$$\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}^n \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

Thank you, C&C Music Factory, for all your contributions. Wait, were they in the Matrix movies? There was that one scene with the house party in Zion National Park or whatever that was, right? Anyway you should probably stop reading this because it has long since stopped being funny. No, really, this isn't going to get any better, and yet here you are, continuing to read this all the way to the end, while you could be solving the next problem. That really is a thing to make you go hmm.

f. Starting with $v(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, see if you can determine $v(3)$ with a very small amount of work.

Important Stuff

2. **a.** The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What's the area of the unit square again?

b. Transform the unit square using the rule

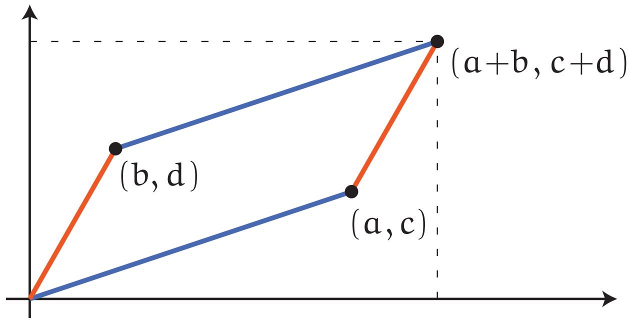
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What are the coordinates of the new shape, and what is its area?

Did you get the correct area? Perfect.

3. Find the area of this parallelogram. Do not continue to the next problem until you talk to someone else at your table who did this a different way.

Hey everybody, did you know that the d—*Andy, get out of here! Shh!*



4. John wants to know the scale factors for the rule

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 8 & -10 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

He doesn't need to know the vectors that get scaled by these scale factors. Help him find the scale factors with minimal algebraic impact.

Don't just make up numbers! John will know if you are lying.

Neat Stuff

5. **a.** Becky multiplies two matrices and gets this:

$$\begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix}$$

Oops, the middle matrix is missing! What is it!

The matrix seems to have no effect on it. It is . . . the one!

b. Benjamin multiplies two matrices and gets this:

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix}$$

Huh, this other matrix is also . . . the one!

Oops, the first matrix is missing! What is it!

Oops, we used "Oops" again.

6. **a.** Give an example of two matrices A and B such that the product AB can be computed but BA cannot.

b. Give an example of two matrices A and B such that both products can be computed but $AB \neq BA$.

c. Give an example of two distinct matrices A and B such that $AB = BA$.

This problem is way more fun if you say it like "mattresses". Some examples of products computed on mattresses are . . . goodnight, everybody!

7. Here's a fun rule: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

a. David has a vector. After using this map, he got $\begin{bmatrix} 23 \\ 36 \end{bmatrix}$. What's his vector?

b. Jennifer has a vector. After using this map, she got $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What's her vector?

I'm trying to free your mind. But I can only show you the door. You're the one that has to walk through it.

c. Jasper has a vector. After using this map, he got $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's his vector?

d. Diana has a vector with variables! After using this map, she got $\begin{bmatrix} a \\ b \end{bmatrix}$. What's her vector?

Really, Diana? Your vector has variables?? Seriously what even.

8. Here's a super fun rule: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

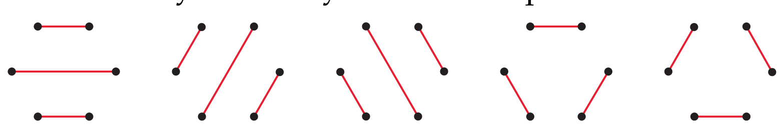
Apply this rule to $\begin{bmatrix} 23 \\ 36 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} a \\ b \end{bmatrix}$.

I'm beginning to doubt the accuracy of the fun level described in these problems.

9. What is the sum of the squares of the numbers in a row of Pascal's Triangle? Any thoughts about why this works?

10. Six people are seated at a round table. Three pairs of them simultaneously shake hands in such a way that their arms don't cross over each other. There are five different ways that they could accomplish this.

Assume everyone at the table has really long arms so that they can always reach across the table.



a. How many possible ways could 4 people simultaneously shake hands like this?

b. . . . 8 people?

c. Based on your work over the last few days, make a conjecture for the number of ways to perform these handshakes for 10 people or more.

11 people: 0 ways.

11. Redraw one of your handshake patterns for 8 people. Pick one of the vertices. (It doesn't matter which, but just be consistent every time you do this.) Write down a **U** because that vertex is connected to a new, unseen chord. Now go around the table clockwise. For each vertex, write down **U** (for unseen) if you meet a new chord you haven't seen before, or **R** (for rerun) if you meet a chord that you've already seen.

If you don't like using **U** and **R**, you can also use "**W**hat is this thing, I can't even..." and "**G**ot this boring thing already."

a. Could this sequence of **U** and **R** ever have more **R** than **U** in it? Explain how you know.

b. Use this to figure out how many possible handshake arrangements there are for $2n$ people.

It may help to do problem 8 from Day 9 if you haven't yet.

12. Hey, remember how you built that big matrix of numbers for the Crhyme schemes? Well . . .

Multiply these two matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & -1 & 6 & -5 & 1 \end{bmatrix}$$

Crhyme schemes often generate opportunities for the one. Remember that part of Matrix where there was that old guy in a chair, and they talked for like 10 minutes about weird boring stuff? Man, what was that. Anyway it was a Crhyme scheme that even made that scene possible. A heist!

Tough Stuff

13. What was that thing you just did in problem 12? Explain why it works.

14. How many combinations are possible on a 5-button Simplex lock? When you push a button on this lock, it stays in place and can't be pushed again. However, you can push more than one button at once, and activate a different combination. Not all 5 buttons need to be pushed in a combination, either. So . . . millions?

This must be one of them Fermi things. The Estimathon is tonight. It's not clear why they needed to know exactly how many people will be coming.

15. For nonnegative integers n, calculate

$$\frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\,dx$$

16. We've been transforming simple shapes (like triangles and squares) with these maps. What happens to other shapes like circles?

It's Free Slurpee Palindrome Day! Go get a Slurpee!!

Day 11: The Matrix Rotations

Opener

1. Here's a recursive rule: $$v(n) = \begin{bmatrix} 7 & -3 \\ 1 & 3 \end{bmatrix} v(n-1)$$

So, if this is like the movies, today is the last day of the Matrix! Sadly the real world is not like the movies, especially not like the Matrix movies' version of the real world.

a. Starting with $v(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

b. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 7 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

c. Find α and β such that $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

These letters are just fancy a and b so if you'd rather use that, do it. Or use emoji! Please do not use emoji as variables, especially not the Samsung cookie emoji. That is not a cookie. Cookie Monster gets super sad when he uses a Samsung phone for this reason.

d. Use your answers to find a closed form expression for $v(n)$. Check to see if it agrees with your computation in part a.

Important Stuff

2. Here's another rule, in case you feel like we're doing the same thing, over and over again:

$$v(n) = \begin{bmatrix} 0 & 1 \\ -30 & 13 \end{bmatrix} v(n-1)$$

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$ compute $v(1), \dots, v(4)$.

b. Determine the vectors that get scaled by this rule, along with their scale factors.

You can do it! To celebrate, we're going to start calling these "I Can" vectors.

c. Write $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$ as a combination of other useful vectors:

$$\begin{bmatrix} 2 \\ 13 \end{bmatrix} = \alpha \begin{bmatrix} \quad \\ \quad \end{bmatrix} + \beta \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Someday you'll find it, the Matrix connection, the vectors, recursions, and you. All of us under its spell, we know that it's probably a malicious computer network . . .

d. Use your answers to find a closed form expression for $v(n)$. Check to see if it agrees with your computation in part a.

3. For a 2-by-2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the *determinant* is $ad - bc$. This also seems to be the product of the two scale factors connected to a matrix.

a. Build a 2-by-2 matrix with determinant 12.
b. If a matrix has determinant 12, what could its scale factors be?
c. Investigate a way to take a 2-by-2 matrix and directly determine the *sum* of its scale factors.
d. Build a 2-by-2 matrix with scale factors 6 and 3. Wow!

Hey everybody, did you know the d—*Andy, we just talked about that!*

4. Here are Kayleigh's three favorite matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

a. Calculate the determinants of A, B, and C.
b. Calculate the products: AB, BA, AC, CA, BC, CB.
c. Calculate the determinants of all the matrices from part b. What do you notice?

The Identity Mattress is so comfortable, you won't even know that it's there!

Nothing! I notice nothing! Good day, sir!

5. Matthew's favorite matrix is $D = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

a. Calculate the products AD, BD, CD, and the determinant of each. What happened?
b. Draw the parallelogram represented by the columns of D. What happened?

Draw the parallelogram in the style of problem 3 from Day 10.

Neat Stuff (feat. Matrices)

6. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2-by-2 *identity matrix*. What do you think it means for a matrix to be an *identity*? Compare to other uses of *identity* in mathematics.

Choose Your Own Adventure! If you'd rather do some counting problems, go to page 46.

Don't tell anybody! Then it can remain a secret identity.

7. Here's a recursive rule: $$v(n) = \begin{bmatrix} -9 & 6 \\ 12 & 5 \end{bmatrix} v(n-1)$$

This matrix has scaled vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with scale factor 9 and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ with scale factor -13.

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, compute $v(1), v(2), v(3)$.

b. Starting with $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$, compute $v(1), v(2), v(3)$.

Look for a way to do the first one with relatively little muss or fuss.

c. Write $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ as a combination of other useful vectors.

d. Starting with $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$, find a closed form expression for $v(n)$.

These "I Can" vectors are really useful!

8. Here's a recursive rule: $v(n) = \begin{bmatrix} 2 & 2 \\ 6 & 1 \end{bmatrix} v(n-1)$

Starting with $v(0) = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$, say "I can do it!", then determine a closed form expression for $v(n)$.

This problem looks familiar, like deja vu. Maybe it's a glitch in the Matrix.

Whoa, you know kung $f(u)$.

9. Using the matrices from problem 4 . . .

a. Calculate the determinant of A^2.
b. Calculate the determinant of $3A$.
c. Calculate the determinant of $\frac{B}{10}$.
d. Calculate the determinant of $A + B$.

$3A$ means to multiply every number of the matrix A by the number 3.

10. **a.** Prove that when you multiply two 2-by-2 matrices, you also multiply their determinants.
b. If two matrices multiply together to make an identity matrix, what can you say about their determinants?

Boy, what *can't* you say about their determinants? They really are high-quality numbers. And definitely numbers, oh yes! No letters, that's for sure. These numbers are terrific.

11. **a.** Find a, c such that $a \begin{bmatrix} 5 \\ 13 \end{bmatrix} + c \begin{bmatrix} 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b. Find b, d such that $b \begin{bmatrix} 5 \\ 13 \end{bmatrix} + d \begin{bmatrix} 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

12. Find a, b, c, d such that

$$\begin{bmatrix} 5 & 8 \\ 13 & 21 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

They say you're supposed to invert your mattress every three months.

Neat Stuff (feat. Counting)

13. There are seven counting problems in this course so far that involve the *Catalan numbers*:

$$C(1) = 1$$
$$C(2) = 2$$
$$C(3) = 5$$
$$C(4) = 14$$
$$C(5) = 42$$
$$C(6) = 132$$

Look through previous problem sets to find them, and try the problems if you haven't had a chance.

The Opener for Day 7 is the first time the Catalan numbers appear. Problem 6 on Day 7 (Crhyme schemes) is the second appearance.

14. In each problem involving the Catalan numbers, you counted a certain set of objects. Since the number of these objects was the same each time, there should be some connection between the objects. Figure out some of the correspondences between these problems. Cool, this is problem 14.

There are 14 ways to stack coins with 4 coins on the bottom, and 14 Crhyme schemes for 4-line poems. So, there should be a way to turn each coin stack into a Crhyme scheme.

Once you find one connection, a new connection to either counts as a connection to both!

15. A *Simplex lock* has five buttons. A combination can involve any or all buttons, and more than one button can be pressed at a time. When a button is pressed, it locks in place and can't be pushed again, so pushing 3, then 4, then 3 is not possible. But pushing 2, then 3 and 4 at once, that's cool.

How many combinations are possible on a 5-button Simplex lock? Billions?

One combo is push 3, then 4, then 5, then enter the pool. Different: push 4, then 3, then 5. Different: push 3 and 4 at once and then 5. Different: push all of 3, 4, and 5 at once. Different . . .

Are there more combinations than square miles in Utah? Only Fermi knows for sure!

Tough Stuff

16. How many combinations are possible on a 6-button Simplex lock?

17. Establish a connection between the Simplex lock and some other set of numbers from this course.

18. The *Tribbiani sequence* is defined by the recurrence

$$f(n) = f(n-1) + f(n-2) + f(n-3)$$

If you start with any set of nonzero integers, the ratio of consecutive terms does . . . what?

The real Tribbiani sequence only has two terms: Friends, Joey.

Day 12: Eigen Do It!

Opener

1. Here's Kristy's favorite recursive rule: $v(n) = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} v(n-1)$

And Oyinka's favorite starting data: $v(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

These two go great together! Time for a high west . . . I mean a high five.

a. Calculate $v(0), \ldots, v(3)$ using the rule, then complete this table. Refer to the Opener (part a) from Day 3 for the values of $J(n)$.

n	0	1	2	3
$v(n)$	$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$			
$J(n)$	2			

b. Determine a closed form expression for $v(n)$ using eigenvalues and eigenvectors. Check that your result gives the correct values for $v(1)$ through $v(3)$.

Use the 2-by-2 matrix to find those pesky "I Can" vectors! Through yesterday we called these *scale factors* and *scaled vectors*.

Important Stuff

2. **a.** The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What's the area of the unit square again?

Stop asking this question, over and over again!

b. Transform the unit square using Kate's rule

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \frac{7}{2} & -3 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What are the coordinates of the new shape, and what is its area?

c. Find the eigenvalues and eigenvectors of the matrix above.

d. Anna wonders what happens to the new shape if you repeatedly transform it using the matrix. So . . . ?

You say, she talks so all the time. So. A palindrome name for palindrome day!

3. Define $v(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ so that Kristy's rule becomes

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

Kristy's rule was mostly benevolent. Mostly.

a. Use matrix multiplication to build this system of equations:

$$\begin{aligned} x(n) &= \qquad\qquad\qquad\quad y(n-1) \\ y(n) &= -10x(n-1) + 7y(n-1) \end{aligned}$$

Since $x(n) = y(n-1)$ for all n, that also means $x(n-1) = y(n-2)$ and $x(n+1) = y(n)$.

b. Take the two equations from part a and combine them into a single equation involving only x or only y. Connect that equation to the Day 3 Opener.

4. Melanie likes the recursive rule

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

This rule is as easy as 0, 1, 2, 3! Okay fine it's 0, 1, 3, 2, whatever.

with starting data $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

a. Say "I can do it!" and find a closed formula for $x(n)$.
b. Compare your work with the Monica sequence.

The Monica sequence was featured in the Opener on Day $C(2)$, where $C(n)$ is the Catalan numbers.

Neat Stuff (feat. Matrices)

5. a. Build a 2-by-2 matrix A so that the equation

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = A \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

models the recursion $x(n) = 2x(n-1) + 35x(n-2)$.

b. Find the eigenvalues and eigenvectors for matrix A.

6. Andrew likes this recursive rule:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Andrew likes tea with his fractions. Tea, frac. Tea that frac! Tea, frac, tea that frac!

a. Calculate the eigenvalues and eigenvectors of the matrix above.

b. Abigail's favorite triangle has vertices $A(1,1)$, $B(1,3)$, $G(3,1)$. Graph this triangle.

c. Transform the triangle three times using Andrew's rule, graphing the new triangle each time. Notice anything?

d. Graph two lines through the origin that show all possible scaled vectors. Investigate how Abigail's triangle changes, relative to the graphed eigenvectors.

You can do it! You've got this problem in the BAG.

7. Try problem 10a from Day 9. Find eigenvalues and eigenvectors of your matrix. Don't be afraid of the complexity!

Just say "i can do it!"

8. Find the eigenvalues and associated eigenvectors for these matrices:

a. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ -6 & 6 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 2 & 9 \end{bmatrix}$

9. If you pick two vectors from the origin, (a, c) and (b, d), the shape made from those vectors (and their sums) is a parallelogram. In three dimensions, the same concept determines a *parallelepiped* spanned by three vectors from the origin: (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

This could be the best math term ever. Parallelepiped! Say it three times: parallelepiperhaps Michael Keaton will come out of the box, or a parallelepied parallelepiper might appear. A die made in this shape has parallelepips on it.

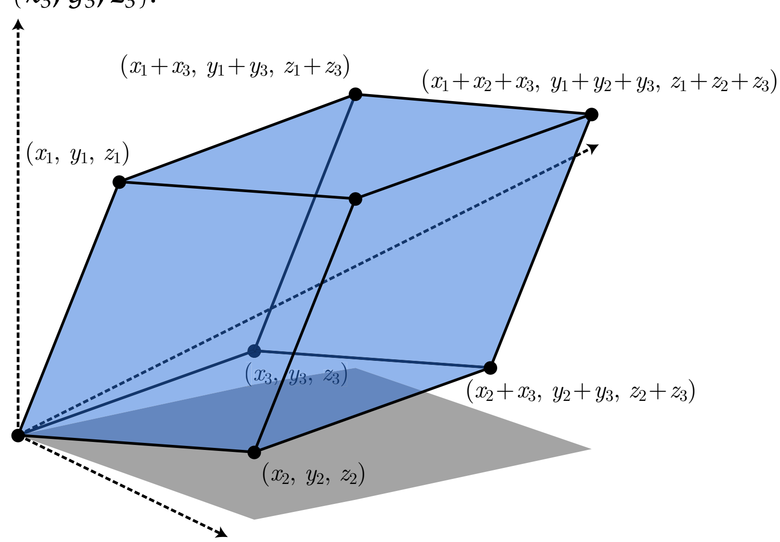


Figure adapted from http://en.wikipedia.org/wiki/File:Determinant_parallelepiped.svg.

Find the volume of the parallelepiped in terms of the nine variables. See if you can do this geometrically, without relying on a formula.

This problem is shear madness.

10. Convert this recursive rule involving a 3-by-3 matrix

$$\begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \\ z(n-1) \end{bmatrix}$$

into a three-term recursion rule for one variable. How might this work for four-term recursions?

11. Yesterday, Bowen made a claim that the final answer for the Opener would be the same if we had written the eigenvector as $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$. Test this by using different eigenvectors in Day 11's Opener.

Bowen also claimed to be in a Kentucky Fried Chicken commercial, and that Rosie O'Donnell called him cute.

12. Let $A = \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}$.

a. Calculate A^2 and A^3. What will A^n be?

b. Consider the recursive rule

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

with starting data $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Instead of finding eigenvalues and eigenvectors, use your work from part a to find a closed form expression for $x(n)$ and $y(n)$.

13. Consider the recurrence relation

$$\begin{bmatrix} a(n) \\ b(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & 2k \end{bmatrix} \begin{bmatrix} a(n-1) \\ b(n-1) \end{bmatrix}$$

with starting data $\begin{bmatrix} a(0) \\ b(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2k \end{bmatrix}$.

Consider yourself, our matrix . . . we don't want to have no muss . . . for after some consideration we can say . . . consider yourself k^2x.

a. How does this relate to problem 18 on Day 5?

b. Make the substitution

$$a(n) = kx(n) + y(n)$$
$$b(n) = k^2x(n) + 2ky(n)$$

so that you no longer have a and b but instead have x and y.

c. Solve the new equations for $x(n)$ and $y(n)$ in terms of $x(n-1)$ and $y(n-1)$.

d. Use the previous problem to find closed form expressions for $x(n)$ and $y(n)$.

e. Reverse the substitution to obtain closed form expressions for $a(n)$ and $b(n)$.

f. Repeat this problem for arbitrary starting data $v(0) = \begin{bmatrix} p \\ q \end{bmatrix}$.

While you're visiting High West, be sure to ask them if you're allowed to order dinner. Chances are nonzero that they'll say no, even though you are seated at a table as a paying customer!

Neat Stuff (feat. Counting)

14. Sunny has flipped directly to the Counting problems, just like she did yesterday. She wants to know the number of different ways to tile a 2-by-10 rectangle with identical 1-by-2 dominoes. Consider any rotations or reflections to be different tilings.

And this is a *good thing*! Choose your own adventure!

15. Mary hates the number 1, and wants to count the number of ways to write 10 as a sum without ever using the number 1. For example:

$$\begin{aligned} 10 &= 8+2 \\ &= 2+8 \\ &= 3+4+3 \\ &= 4+3+3 \\ &= 3+3+4 \\ &= 10 \\ &= 2+2+2+2+2 \\ &= 5+5 \\ &= 2+6+2 \\ &= \cdots \end{aligned}$$

How many different ways can Mary write 10 like this?

Nobody tell her that there's a "1" inside the number 10, okay?

16. Becky built a chart to deal with the Simplex lock.

	# of distinct pushes					
	0	1	2	3	4	5
0-button lock	1					
1-button lock	1	1				
2-button lock	1		2			
3-button lock						
4-button lock		15				
5-button lock						

How many different combinations are there on a 5-button Simplex lock?

See Day 11 for a description of the Simplex lock. Okay, the zero-button lock and the zero-button combination are not very useful, you'll have to trust us on this one.

17. Use problem 10 from Problem Set 7 to establish this relationship among Catalan numbers:

$$C(n+1) = \sum_{i=0}^{n} C(i)C(n-i) \qquad \text{for } n \geqslant 0$$

with the starting data $C(0) = 1$. For instance, if $n = 3$,

$$C(4) = C(0)C(3) + C(1)C(2) + C(2)C(1) + C(3)C(0)$$

This problem is proof that Catalan numbers can march up and down, and definitely belong in a parade.

18. Use your work from Day 8 to establish this closed form expression for Catalan numbers:

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Tough Stuff

19. Of all the combinations on a 5-button Simplex lock, how many of the combinations involve pressing all five buttons, and how many involve less than five buttons? Hmmm! Looks like a proof might be useful here.

Hm. Is this problem simple, or is it complex? Or perhaps it is both. Simplex!

20. Find some recursive rules that are satisfied by $S(n)$, the total number of combinations on an n-button Simplex lock. Then find the number of combinations on a 10-button Simplex lock.

Day 13: Prime of My Life

Opener

1. Sean likes this recursive rule:

$$v(n) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} v(n-1)$$

Now I've had the prime of my life, no I've never factored this before. Yes I swear, I have proof, and I owe it all to you . . . clid.

a. Ariel's favorite triangle is $F(1,1)$, $L(2,1)$, $Y(2,3)$. Graph this triangle.

b. Transform the triangle five times using Sean's rule, graphing the new triangle each time. What do you notice about the triangle's location and coordinates?

Just like the closing dinner, this triangle seems to have unlimited rolls!

c. Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

d. Find a closed form expression for $v(n)$ if

$$v(n) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} v(n-1)$$

with starting data $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

It's tricky to rock these eigs, to rock this problem right on time, it's tricky! How is it, D?

Important Stuff

2. Compute the area of each triangle in problem 1. Cool!

Sean says that if you stare at these triangles long enough, it'll show a 3-D magic eye pattern.

3. Gabe's favorite matrix is $\begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix}$.

a. Find eigenvalues and eigenvectors for Gabe's matrix.

b. Chris loves to cancel out Gabe! Find his matrix, the one that makes this equation true:

$$\begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Careful, Gabe, or Chris will tell you the winner of The Genius Season 3 before you can watch it! Here's a hint: it's not Doohee.

c. Find eigenvalues and eigenvectors for Chris's matrix. Oh, snap!

4. Here's Gabie's recursive rule: $v(n) = \begin{bmatrix} -13 & 60 \\ -5 & 22 \end{bmatrix} v(n-1)$

a. Find eigenvalues and eigenvectors of the matrix in her rule.

b. Gabie is interested in the ratio between the top and bottom numbers in the vector $v(n)$, as n increases. She starts with $v(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Use what you've learned about eigenvalues and eigenvectors to predict what that ratio will be as n increases.

c. Calculate $v(1), \dots, v(5)$ by matrix multiplication (with or without techmology), and calculate the ratio of the top to bottom number in each vector. What happens?

Calculating for so long, now I finally found a 1 coprime to me . . . saw the writing on the wall, it's one of those vertical surfacies . . .

For example, $v(1) = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$, and the ratio is about 1.67.

Review Your Stuff

5. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 15 minutes** on this. Make sure your question is something that $\boxed{*\textbf{everyone}*}$ at your table can do, and that you expect $\boxed{*\textbf{everyone}*}$ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. Just say "I can do it!" Then actually do it.

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff (feat. Matrices)

6. Two lines are perpendicular at the origin. The sum of their slopes is 1. What are the equations of the lines?

7. Find eigenvalues and eigenvectors of these matrices.

a. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ **c.** $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Huh. That is worrisome...

You're the vector . . . I take multiples of. Now all this fancy math . . . fits like a glove! Because . . .

They try to make me act like other matrices, and I say no, no, no.

8. **a.** Find a, b, c, d to make this equation true:

$$\begin{bmatrix} 3 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity, and two matrices that multiply to an identity are inverses.

b. Find e, f, g, h to make this equation true:

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Find a general formula for the inverse of a 2-by-2 matrix. In other words, find a matrix that makes this equation true:

See problem 2 from Day 6, problem 5 from Day 10, problem 6 from Day 11.

The symbol A^{-1} denotes the inverse of a matrix A.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

d. What must be true about a 2-by-2 matrix for it to have an inverse?

9. What must be true about a, b, c, d so that the system of equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

has exactly one solution for x and y?

10. Andy tells you the formula for the determinant of a 3-by-3 matrix is

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = x_1y_2z_3+x_2y_3z_1+x_3y_1z_2-x_3y_2z_1-x_2y_1z_3-x_1y_3z_2$$

Hey everybody, did you know the determinant of a 3-by-3 matrix is also the vol—*yes Andy, didn't you see the cool picture on Day 12?*

Calculate the determinant of the matrices in problem 8 from Day 12. Are the determinants related to the eigenvalues in any way?

11. Think about Gabie's experiment in problem 4. Find a starting vector, besides $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, that does *not* eventually lead to the same ratio of top to bottom numbers.

Every vector dance now! Every vector dance now! You better be ready, math nerds!

12. Here's another of those charts! This one gives the coefficients of expanded polynomials coming from $(x-1)(x-2)\cdots(x-k)$. Complete the chart and look for some recursions in it.

	coefficient of power of x					
terms	x^0	x^1	x^2	x^3	x^4	x^5
1	1					
$x-1$	-1	1				
$(x-1)(x-2)$	2	-3				
3 terms		11				
4 terms	24				1	
5 terms						

With imaginary i's there's no way we can disguise complexity . . . so we take each others' hand, trying to help us understand knot theory . . . just remember! You're the 1 thing, making identities . . . some of this matrix stuff, leads to obscenities!

The "3 terms" are $(x-1)(x-2)(x-3)$. Look, this one goes to 11!

13. Hm, both the chart in problem 12 and the chart of rhyme schemes from Day 6 are actually 6-by-6 matrices, if you let all the gray boxes be zeroes. I wonder what happens when they are multiplied.

14. You know, that means the chart of Crhyme schemes from Day 7, the one that generated Catalan numbers, is also a matrix that may have an inverse. (Oh it does.) Determine the inverse matrix and look up and down for some patterns in it.

If someone catches you inverting a matrix, just say it wasn't you. As long as you're not doing that on a bathroom floor, it should work.

15. Let's do one more chart! This one gives the coefficients of expanded polynomials coming from

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k)}{k!}$$

Complete the chart and look for some recursions in it.

Let's do this one more time. Oh oh oh. Can't stop, we're higher than a Mount Washington.

	coefficient of power of x					
	x^1	x^2	x^3	x^4	x^5	x^6
$\binom{x}{1}$	1					
$\binom{x}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$				
$\binom{x}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$				
$\binom{x}{4}$		$\frac{11}{24}$	$-\frac{1}{4}$			
$\binom{x}{5}$	$\frac{1}{5}$				$\frac{1}{120}$	
$\binom{x}{6}$						

Charts, charts, charts, charts charts charts! EVERYBODY!

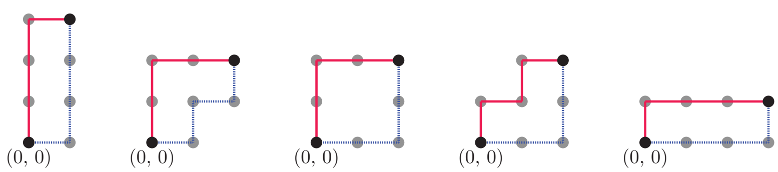
The Hsu Shay Resort would like to announce that all skiing "shoosh" sounds are being replaced by "shoop" sounds. Shoop Hsu Shay Shoobie, like Scoobie Doobie.

Dang, this matrix probably has an inverse, too. Alright, let's find the inverse.

Neat Stuff (feat. Counting)

16. Sam and Roy start at the origin in the plane. Sam moves up one unit and Roy moves to the right one unit. From that point on, each of them can either move up or right, but they don't run into each other again until they have moved exactly four spaces (including the first move). There are five different ways that they could do this.

Ohh, one two, hikers stand before you, that's what I said now, each one gonna take a different path now, just go ahead now. Neither has diamonds in his pockets, but they've got bread, now.



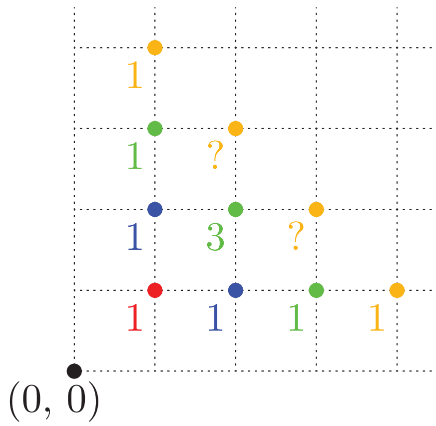
a. How many different ways could they do this if they each move exactly three spaces?

b. . . . five spaces?

c. Connect this problem back to work from previous problem sets.

17. Let's revisit Sam and Roy's problem from a different perspective. Given the constraints of how Sam and Roy are able to move, how many different ways can they meet up at each of the lattice points?

Ask Sean, maybe this is a magic eye 3-D question. That purple shirt, though.



Aw, this is going to be another one of those charts,isn't it.

a. This triangle of numbers connects to many of the problems we've seen (Crhyme schemes, Grey/White games, non-cheese paths, etc.). Find as many connections as you can.

b. Does this triangle of numbers build in a way reminiscent of Pascal's triangle? Figure stuff out!

The Sam-Roy dance, is your chance, to do some math. You know what we're doin', we're doing the Sam-Roy math.

Tough Stuff

18. Determine the value of this sum without the use of techmology.

$$\sum_{k=0}^{\infty} \frac{k^3}{k!} =$$

Extend to other powers of k.

19. Explain why the matrix inverse relationship between rhyme schemes and products of $(x-1)(x-2)(x-3)\ldots$ exists.

I'm pretty sure it exists for the sake of problems like this one.

20. Find a 3-by-3 matrix A so that A^{12} is the identity matrix but no earlier power A^k with $k > 0$ is the identity. No cheating by rigging up a 2-by-2 matrix with an extra "1" in the corner.

That's 1 in the corner. That's 1 in the spotlight, losing a dimension.

Day 14: Crazy Ex Math Camp

Opener

1. Given this ski resort:

This problem courtesy of Table 1!

a. Fill in this matrix representing the number of ways to travel between locations using exactly one token.

... to

# ways from	A	B
A		
B		

b. Find the matrix representing the number of ways to travel between locations using exactly *ten* tokens. Bonus to those who can figure out how to do this without techmology!

c. If you start at B, what happens to the ratio of the number of ways to get to B to the number of ways to get to A as the number of tokens gets larger?

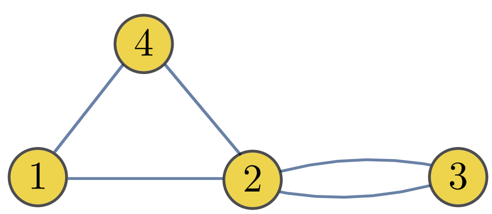
Hey everybody, did you notice that A and B from the Hsu Shay Resort were set up this—*oh, that's actually pretty interesting, Andy.*

Important Stuff

2. G is the graph ...

All paths go in both directions for this graph.

Oh the math of love triangles . . . is as simple as can be! Whichever Tom or Dick, I might pick, the center of the triangle is little ole me! (Actually, a triangle has multiple centers.)



a. Find the adjacency matrix A.

b. Find the matrix giving the number of three-step walks.

Do you like apples?

3. **a.** Vicki loves the matrix $A = \begin{bmatrix} 4 & 8 \\ 7 & 5 \end{bmatrix}$ and uses it in a recursive rule $v(n) = Av(n-1)$ with starting vector $v(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $v(1), v(2), v(3), v(4)$.

This is a terrible scandal, and we're sorry we had to reveal it in this way.

b. Will is impatient and wants to get bigger numbers faster so he defines $B = A^2$. Calculate B.

This will get him time with the baby faster.

c. Will uses B in a recursive rule $w(n) = Bw(n-1)$ with starting vector $w(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $w(1)$ and $w(2)$.

d. Calculate eigenvalues and eigenvectors for A and B. Use these to write closed form expressions for $v(n)$ and $w(n)$. Notice anything?

4. Troy likes Monica's two-term recursive rule from Day 2

$$\begin{aligned} M(0) &= 2 \\ M(1) &= 2 \\ M(n) &= 2\,M(n-1) + 3\,M(n-2) \quad \text{if } n > 1 \end{aligned}$$

Troy's sequence skips over Monica's in this way: $T(0) = M(0)$, $T(1) = M(2)$, . . . , $T(10) = M(20)$, etc. Skips, skips, skips, skips skips skips! EVERYBODY!

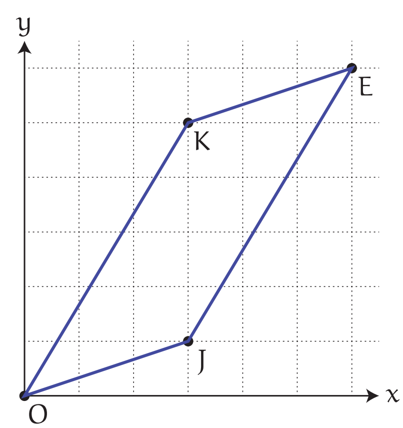
but he's impatient and wants to get bigger numbers faster. Write a new two-term recursive rule for $T(n)$ that skips every other term in Monica's sequence.

Your Stuff

Your jokes. *Our jokes.*

T2. Find a matrix that takes the shape shown to the unit square.

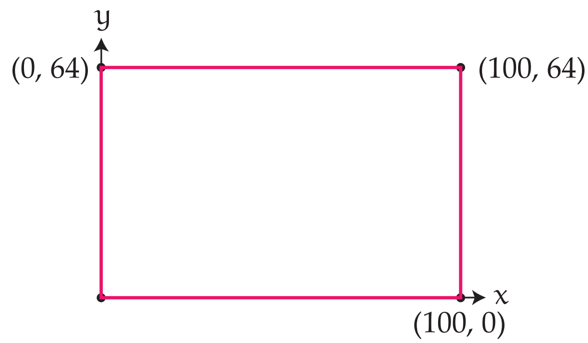
Look around for the joke.



Oh, the math of love parallelograms . . . has a simple little trick. To find the area of the parallelogram . . . just use the theorem of Mr. Pick! (Actually that's only true for lattice points.)

T11. Huey says "It's hip to be a rectangle," in a bid to gain more pop cultural relevance. He decides to transform all the unit squares in his condo to rectangles. One of this transformations looks like this:

It's really more hip to be a triangle, now. Huey should run for governor of Louisiana in a bid to gain more literary relevance.



a. Find a 2-by-2 matrix that will help Huey transform the unit square appropriately.

b. Huey isn't a "math person." Find two different matrices that can accomplish the same result with smaller, Huey-friendly entries.

Huey really just has a very poor mindset. He believes in the power of love, but not powers of matrices.

T5. **a.** Create a 2-by-2 matrix with eigenvalues 3 and 4.

b. Create a 2-by-2 matrix with one distinct real "I Can" value.

c. Does this matrix still create a parallelogram when starting with a unit square?

T6. Given $v(n) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} v(n-1)$ with $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

You know nothing, Jon Snow.

a. Find $v(1), v(2), v(3)$. Graph each.

b. Describe the transformation. Be specific.

c. How many mappings are necessary to get back to the original vector?

That's right! Going left is descriptive. Going left 12 units is specific.

T7. Sam's favorite triangle is $S \begin{bmatrix} 2 \\ 8 \end{bmatrix}, A \begin{bmatrix} 0 \\ 6 \end{bmatrix}, M \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

a. Find the area of SAM.

b. Sam's arch enemy triangle is $J \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E \begin{bmatrix} -2 \\ -2 \end{bmatrix}, N \begin{bmatrix} 6 \\ -7 \end{bmatrix}$.

Graph and find the area of JEN. Notice anything?

I always thought the arch enemy was Burger King.

c. Find the determinant of $\begin{bmatrix} -2 & 6 \\ -2 & -7 \end{bmatrix}$. What does this represent geometrically? What connections do you see?

T8. Melanie likes the recursive rule

$$v(n) = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} v(n-1)$$

a. Find the eigenvalues and eigenvectors for Melanie's matrix.

b. Melanie's favorite triangle is $I \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $D \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $K \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Graph IDK.

c. Transform IDK five times using Melanie's rule, graphing the new triangle each time. What do you notice about the triangle's location and coordinates?

d. What happens if Melanie keeps applying her rule to IDK?

Imagine i, a matrix . . . You may say I'm a vector and I'm having fun. I hope some day you'll scale me, and this problem will be done.

Wait, I missed that. What triangle was it? IDK.

All matrix work and no play make Melanie something something.

T4. The inverse of matrix A is $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Without directly finding A, write a closed form expression for $v(n) = A^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

T9. **a.** Solve for k: $0 = k^2 - 8k + 15$

b. Solve for k: $0 = 15k^2 - 8k + 1$

c. Take the quadratic from part b, substitute in $1/m$ for k and solve for m. What do you notice?

d. Given MAD GT's favorite quadratics

$$0 = ax^2 + bx + c$$
$$0 = cx^2 + bx + a$$

what can you say about the solutions?

MAD GT is the name of Table 9's biker gang. Find them at the Cabin on Thursday nights. Don't get them confused with Happy GTs . . . they don't like being confused with the Happy GTs.

T12. Here's a three-term recursive rule:

$$F(0) = 0, F(1) = 1, F(2) = 1$$
$$F(n) = F(n-1) + F(n-2) + F(n-3) \quad \text{if } n \geqslant 3$$

a. Find the first 10 terms.
b. What patterns do you notice?
c. If you were going to find a closed form, what types of equations might you need to work with/solve?
d. Explore as desired.

Go cube-ic!

T3. Given a recursive rule

$$v(n) = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} v(n-1)$$

with starting data $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the closed rule

$$v(n) = a\left(\quad\right)^n \begin{bmatrix} \\ \\ \end{bmatrix} + b\left(\quad\right)^n \begin{bmatrix} \\ \\ \end{bmatrix}$$

What is the general rule for the magic farm animals matrix $\begin{bmatrix} 0 & 1 \\ \text{cow} & \text{duck} \end{bmatrix}$?

T10. **a.** A volleyball team has 1 setter, 2 attackers, and 3 diggers. A sequence of hits must follow the following rules to win a point.

- The ball must be hit in this order: digger $\implies$ setter $\implies$ attacker.

Assuming players are equally likely to hit the ball to anyone, including themselves, find the probability of winning a point given the necessary sequence.

Hey everybody, I thought I told you no more volleyb—*let it go, Andy, let them play!*

b. Now we're playing a new Top Gun version of volleyball, which has the following rules.

- If a person hits a ball twice in a row, they instantly lose.
- You can, but you don't *have* to, have four hits on one side.
- To win a point, some set of three hits in a row must go digger $\implies$ setter $\implies$ attacker.

Now what's the probability of winning?

c. For Top Gun II (coming in 2019) you can have five hits on one side. What's the probability of winning?

Also rules-shirts off, second hit must go to Goose.

No more Goose—he was ejected.

To actually do this as a math problem you will have to ignore the previous two notes. Also Top Gun II is real.

T4. BØW3N is the hottest new AI developed by the Skynet Working Group specializing in dropping mad random rhymez. What are the odds that its randomly generated

10-line poem matches the rhyme scheme of this no-longer-copyright-infringing masterpiece?

Just a small town girl
Livin' in a Prospector
She took the midnight bus goin' to Main Street
A Park City boy
Raised here nowhere near Detroit
He took the midnight bus goin' to Main Street
A cowboy on a smoky stage
No one even knows his age
For a dollar they can share a song
That goes on, and on, and on, and on

Oh, the movie never ends but we're from the days of VHS so you can rewind to Day 6.

Bonus: make a karaoke-worthy randomly generated song. *Done. Oh, whoops, we changed yours.*

Neat Stuff (feat. Matrices)

5. Calculate eigenvalues and eigenvectors for these matrices. Oh, and their determinants too.

So, you want me to determine it?

a. $\begin{bmatrix} -1 & -6 & -3 \\ -9 & -4 & -8 \\ -9 & 7 & -4 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 & 4 & -5 \\ -4 & 1 & -5 \\ -2 & -4 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} -1 & 1 & -2 \\ 3 & -6 & -3 \\ 2 & 1 & -5 \end{bmatrix}$

6. The *diagonal* of a square matrix goes from top left to bottom right. The *trace* of a square matrix is the sum of numbers on its diagonal.

The other diagonal is not a diagonal! Deal with it!

a. How is the trace useful when finding eigenvalues of 2-by-2 matrices?
b. Calculate the trace of each of the matrices above. Calculate the sum of the eigenvalues for each of the matrices above.

7. What happens to the determinant of a matrix when you perform these operations below? Make up some sample matrices to play with or use the ones above.

Hey everybody, did you know the determinant is a function on the TI—*it's a function on all the TI's, Andy!*

a. Swap any two rows in a square matrix.
b. Swap any two columns in a square matrix.
c. Multiply an entire row or column by 10.
d. Multiply the entire matrix by 10.

e. Replace the first row of the matrix by the sum of the original first and second rows.

f. Replace the first row of the matrix by the sum of the original first row and 10 times the second row.

8. In problem 3 on Day 13 you noticed the behavior of eigenvalues and eigenvectors for an inverse matrix.

a. Describe why you think this might always happen.

b. Under what circumstances will a matrix *not* have an inverse?

9. Multiply these. What?

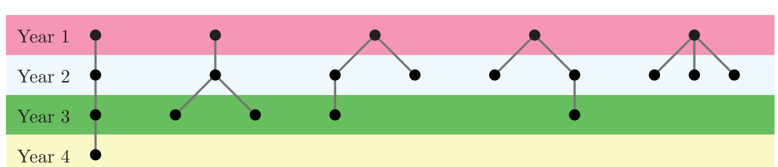
Oh, the math of love mattresses . . . is super super fun! Just multiply on two mattresses . . . and you will finally find the one! (Actually that only works for inverse matrices.)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 6 & 10 & 1 & 0 \\ 0 & 0 & 1 & 20 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 & 0 \\ -17 & 17 & -6 & 1 & 0 & 0 \\ 152 & -152 & 54 & -10 & 1 & 0 \\ -1943 & 1943 & -691 & 130 & -15 & 1 \end{bmatrix}$$

Neat Stuff (feat. Counting)

10. In Year 1 of PCMI, the Awesome Lessons Working Group made an Awesome Lesson that was so awesome that in later years, the group didn't make new lessons. Instead they chose to tweak the previous' year's versions into any number of variations, or just stop.

But you didn't have to come to Utah
Meet some friends and do recursion and learn 'bout eigenvalues
I guess you've got to leave us though
Now we're just some math camp that you used to know



Suzanne found that the working group created four lessons, including the original Awesome Lesson. There are exactly five ways in which the original Awesome Lesson could have evolved, shown above.

a. Suzanne found a different Awesome Lesson chain with 3 lessons. How many different ways could this Awesome Lesson have evolved?

b. Suzanne found another different Awesome Lesson chain with 5 lessons. How many different ways could this Awesome Lesson have evolved? Break down this total into categories, given that the working group could have been doing this for 1, 2, 3, 4, or 5 years.

c. Catalan this sucker. Relate this problem to others.

11. Mary is back, and doesn't hate the number 1 like she did on Day 12. But now she hates all even numbers! She wants to count the number of ways to write numbers as sums without ever using an even number. For example:

Good thing for her this is problem 11. Or problem 23. Either way, she's good.

$$\begin{aligned} 10 &= 7 + 3 \\ &= 3 + 7 \\ &= 7 + 1 + 1 + 1 \\ &= \cdots \end{aligned}$$

How many different ways can Mary write the number n like this?

Now and then I think of all the times you gave me Neat Stuff
But had me believing it was always something I could do
Yeah I wanna live that way
Solving problems on a Saturday
But now you've got to let us go
And we're leaving from a math camp that you used to know

Tough Stuff

12. How many ways are there to tile a 2-by-n rectangle with rectangles of integer side lengths?

13. What percentage of the time will a Fibonacci number $F(n)$ and its corresponding Catalan number $C(n)$ share a common factor greater than 1?

14. Each of the entries of an n-by-n matrix A are randomly chosen (with equal probability) to be a 0 or 1. What is the probability that A has an inverse? What happens to that probability as n goes to infinity?

No More Stuff

15. Thanks. We had a wonderful time and hope you did too. See you again as soon as possible.

We miss you so, so bad. See you on the next exciting episode of *The Gone Show*! Oh wait, we're canceled.