Day 1: Over + Over Again

Welcome to PCMI! We know you'll learn a great deal of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about eigenvectors, think again: the problems should lead to the appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That's* why it's Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

PCMI participants have solved at least two previously unsolved problems presented in these courses.

When you get to Day 3, come back and read this again.

Consider this your first excursion into recursion . . .

Opener

1. We're going to start with doing the same thing, over and over. The *Fibonacci sequence* is one of the most famous sequences of all time. It starts with 0, then 1, then each new term is the sum of the two that come before it. A more formal definition is

$$F(0) = 0$$
$$F(1) = 1$$
$$F(n) = F(n-1) + F(n-2) \quad \text{if } n > 1$$

For example, $F(2) = F(1) + F(0)$, then $F(3) = F(2) + F(1)$, then . . . Use the values of $F(0)$ and $F(1)$ to find $F(2)$, then use the values . . . then they tell two friends, and . . .

a. For the Fibonacci sequence, determine $F(0)$ through $F(9)$ and the sum of these ten numbers.

b. Your table will be given four new pairs of starting numbers. For each pair, determine the first ten numbers (including the two givens) and their sum. Notice anything?

c. Describe some similarities between the five sequences your table worked with.

Most of the time if we use $F(n)$ with capital F, we mean the "real" Fibonacci sequence, not these impostors.

Stuff in boxes is more important than other Important Stuff!

Important Stuff

2. Traci defines the sequence $0, 1, 2, 3, 4, \dots$ recursively:

$$t(0) = 0$$
$$t(n) = t(n-1) + 1 \quad \text{if } n > 0$$

a. For some number a, $t(a) = 23$. Find a.

b. Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(9)$.

c. Calculate the sum $t(0) + t(1) + t(2) + \cdots + t(100)$.

3. Write a recursive definition for $a(n)$ that fits the sequence $2, 6, 10, 14, 18, \dots$

This means $a(0)$ should be 2, $a(1)$ should be 6, and $a(73)$ should be 294. Just sayin'.

4. Write a recursive definition for $b(n)$ that fits the sequence $2, 6, 18, 54, 162, \dots$

5. Without a calculator, *estimate* the number of digits in $F(100)$, a big Fibonacci number. Yes, it's fine to get this wrong! But think it over a bit.

Avoid saying "the 100th Fibonacci number" unless it's clear what you mean. $F(100)$ is usually called the 100th Fibonacci number, but it can be confusing.

6. Water you going to drink a lot of today?

7. Find two numbers with the given sum s and product p.

a. $s = 7, p = 10$	**e.** $s = 10, p = 23$
b. $s = 2, p = -3$	**f.** $s = 10, p = -1$
c. $s = -13, p = 30$	**g.** $s = 100, p = 2379$
d. $s = 10, p = 25$	**h.** $s = 100, p = 2017$

Neat Stuff

8. Which Fibonacci numbers are even, and which are odd? Explain why this happens.

Some of the Fibonacci numbers can't even.

9. Which Fibonacci numbers are multiples of 3? Explain why this happens.

10. Naira's favorite sequence starts with $N(0) = 7$ and $N(1) = 4$. After that, each term is the opposite of the sum of the previous two terms. Write the first ten terms of this sequence.

We heard her call this the "neganacci" sequence.

11. The *Lucas sequence* is like the Fibonacci sequence, except it starts with 2 and 1 instead of 0 and 1:

$$L(0) = 2$$
$$L(1) = 1$$
$$L(n) = L(n-1) + L(n-2) \quad \text{if } n > 1$$

Find as many relationships as you can between the numbers in the Lucas sequence and the numbers in the Fibonacci sequence. Try to prove them!

$L(2) = 3, L(3) = 4, L(4) = 7$. There's a lot of literature on Fibonacci and Lucas. We humbly request that you not read any of it for now, so that you have the chance to find and prove some of the results on your own.

12. Ramona's sequence is the sum of the Lucas and Fibonacci sequences.

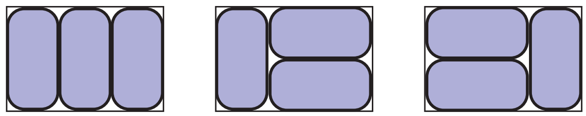
$$R(n) = L(n) + F(n)$$

Figure out what you can about Ramona's sequence, and any new relationships you can figure out between the Lucas and Fibonacci sequences.

She likes this because $L + F = R$ when you're counting letters as money!

13. Without a calculator, determine the units (ones) digit of $F(100)$.

14. In terms of n, how many ways are there to tile a 2-by-n rectangle with identical 1-by-2 dominoes? Consider any rotations or reflections to be *different* tilings: there are 3 tilings for the 2-by-3 rectangle. Why look, here they are!!



15. Carla wrote this sequence for $c(n)$: $1, 2, 11, 43, 184, 767 \ldots$ Find a recursive rule that could define Carla's sequence.

16. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 1 + \frac{1}{r(n-1)} \quad \text{if } n > 0$$

17. Some pairs of Fibonacci numbers $F(a)$ and $F(b)$ have common factors. Investigate and find something interesting about it.

Well, duh, they have the common factor 1. (We mean "legitimate" common factors.)

Tough Stuff

18. Genevieve claims that starting with $F(7) = 13$, it's possible for $F(n)$ to be prime, but it's *never* possible for $F(n) + 1$ or $F(n) - 1$ to be prime. Prove this . . . well, if it's true . . .

19. Find x if

$$\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}}} = 15$$

20. Consider the unit circle $x^2 + y^2 = 1$. Plot n equally spaced points on the circle starting from $(1, 0)$. Now draw the $n - 1$ chords from $(1, 0)$ to the others. What is the product of the lengths of all these chords?

21. Take the diagram you drew in problem 20 and stretch it vertically so that the circle becomes the ellipse $5x^2 + y^2 = 5$. All the points for the chords scale too. What is the product of the lengths of all *these* chords?

What.

Starting Pairs:

A (0, 5)
B (2, 2)
C (2, 7)
D (4, 9)

Starting Pairs:

K (0, –4)
L (2, 1)
M (2, –3)
N (4, –2)

Starting Pairs:

G (0, 3)
H (3, 4)
I (3, 7)
J (6, 11)

Starting Pairs:

P (0, –10)
Q (5, 8)
R (5, –2)
S (10, 6)