Day 3: Mind ÷ Matter

Opener

1. We're going to start with doing the same thing, over and over. Here's a recursive definition for a function $J(n)$.

$$J(n) = 7 \cdot J(n-1) - 10 \cdot J(n-2)$$

Oyinka picks the starting data $O = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and calculates.

a. Determine $J(0)$ through $J(7)$ with Oyinka's starting data.

b. Oyinka says that her $J(n)$ grows exponentially. Is she right? Is she almost right?

c. Beth gives you four new sets of starting data. For each, determine the first eight numbers (including the two from the starting data). Notice anything?

d. Find a closed rule for Oyinka's $J(n)$, then use it to compute $J(8)$ directly.

This means that $J(0) = 2$ and $J(1) = 7$. To find $J(2)$, follow the recursion: $J(2)$ is 7 times $J(1)$, minus 10 times $J(0)$. The value of $J(2)$ is a very common age on dating profiles.

A *closed rule* is one like $J(n) = 3^n + (-1)^n$. It has no recursion, and it also has no recursion.

Important Stuff

2. Find two numbers with the given sum s and product p.

a. $s = 7, p = 10$
b. $s = 2, p = -3$
c. $s = 3, p = -10$
d. $s = 9, p = 14$
e. $s = 8, p = 15$
f. $s = 100, p = 2451$
g. $s = 200, p = 9991$
h. $s = 1, p = -1$

A similar problem was on Day 2's Important Stuff. If you haven't had a chance at that one, try it first, or don't, it's fine, I'm just a piece of paper.

3. The *common ratio* between two terms in a sequence is calculated by dividing a term by the one before it. Calculate the common ratios of Oyinka's $J(n)$, from $J(1)/J(0)$ up to $J(8)/J(7)$, to four decimal places. What up with that?

What up with that, I say, what up with that?!

4. This is Gareth's favorite recursive rule:

$$G(n) = 3G(n-1) + 10G(n-2)$$

Lynde picks the starting data $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

a. Determine $G(0)$ through $G(7)$ with Lynde's starting data.

b. Calculate the common ratio of consecutive terms. What's happening!!

Hey *hey* hey!

5. Angelina uses the recursion from problem 4 but she chooses starting data $\begin{bmatrix} 1 \\ k \end{bmatrix}$. Oh noes! k is a variable!

If the Earth loses its protection from ultraviolet radiation, it will be the fault of the oh noes layer.

a. Pick a number to use for k and determine $G(0)$ through $G(3)$. Do you think this function is exponential? Explain how you know.

b. Determine $G(0)$ through $G(3)$ in terms of k.

c. Find all possible values of k for which the starting data $\begin{bmatrix} 1 \\ k \end{bmatrix}$ produces an exponential function.

6. Find a closed rule for each of the four sets of starting data Beth gave you for $J(n)$ in the Opener.

7. There's a shorthand for the adding and scaling of starting data we've been doing:

Strawberry shorthand is the best flavor.

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad \text{and} \quad 2\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

a. Find A and B so that $A\begin{bmatrix} 1 \\ 5 \end{bmatrix} + B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 19 \end{bmatrix}$.

b. A sequence starts with $\begin{bmatrix} 5 \\ 19 \end{bmatrix}$ and follows the rule $J(n) = 7\,J(n-1) - 10\,J(n-2)$. Find a closed rule for this sequence.

Right now, hopefully you don't feel like it's 5:19 am.

c. Find A and B so that $A\begin{bmatrix} 1 \\ 5 \end{bmatrix} + B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

d. Find A and B so that $A\begin{bmatrix} 1 \\ 5 \end{bmatrix} + B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

e. Find A and B so that $A\begin{bmatrix} 1 \\ 5 \end{bmatrix} + B\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}$. Use the last two!

Neat Stuff

8. There is a two-term recursive definition for $W(n)$ that fits the function $W(n) = 7^n - 2^n$. The rule is

$$W(n) = s \cdot W(n-1) + p \cdot W(n-2)$$

and s and p need to be found. To find s and p . . .

a. Compute $W(0)$ through $W(4)$.

b. Here's a system of two equations

$$W(2) = s \cdot W(1) + p \cdot W(0) \qquad \text{and}$$
$$W(3) = s \cdot W(2) + p \cdot W(1)$$

Solve the system to find s and p.

c. Verify that your recursive definition gives the correct values of $W(0)$ through $W(4)$.

Two-terming could also be called two-timing, because it goes back in time twice. Just like Marty McFly!

Psst: $W(0) = 0$ and $W(1) = 5$.

9. **a.** Find a two-term recursive definition for $X(n)$ that fits the function $X(n) = 3^n + 5^n$.

b. Find a two-term recursive definition for $Y(n)$ that fits the function $Y(n) = 2 \cdot 3^n + 3 \cdot 5^n$.

c. Find a two-term recursive definition for $Z(n)$ that fits the function $Z(n) = 4 \cdot 3^n - 5^n$.

Two-terming is something presidents do, sometimes. But not always.

10. This is Hilda's favorite recursive rule:

$$H(n) = 5H(n-1) - 6H(n-2)$$

Find a closed rule for the sequence that fits each of these starting data.

a. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ **b.** $\begin{bmatrix} 6 \\ 13 \end{bmatrix}$ **c.** $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

11. What happens to the Fibonacci sequence if only units digits are considered? The sequence begins

$$0, 1, 1, 2, 3, 5, 8, 3, 1, \ldots$$

Some might say you are looking at things in "mod 10". Ignore those people and just do the problem.

12. Melvin wonders what happens when you take the product of two Fibonacci numbers that surround a third. What is it?

It's it. What is it? It's it.

13. In terms of n, how many ways are there to write n as the sum of ones and twos? Consider any reorderings to be *different* ways. There are three tilings, uh, ways to write 3 using ones and twos: $1+1+1$ or $1+2$ or $2+1$.

14. In terms of n, how many binary sequences of length n do not have consecutive zeros?

A *binary sequence* is made up of all ones and zeros. For $n = 2$ there are four binary sequences: 00, 01, 10, and 11.

15. Describe what happens with the sequence defined by

$$r(0) = 1, \qquad r(n) = 7 + \frac{-10}{r(n-1)} \quad \text{if } n > 0$$

Repeat for $r(0) = 2$. Neato.

16. Closed rules, closed rules!

Come get these *closed rules*! I know you want it (hey hey hey).

a. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ but this time for $t(n) = 5\,t(n-1) + 6\,t(n-2)$

b. $\begin{bmatrix} 2 \\ 100 \end{bmatrix}$ for $t(n) = 100\,t(n-1) - 2451\,t(n-2)$

c. $\begin{bmatrix} 2 \\ 10 \end{bmatrix}$ for $t(n) = 10\,t(n-1) - 23\,t(n-2)$

d. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for $t(n) = t(n-1) + t(n-2)$

The Lucas sequence was once injured in a high school football game, but went on to star in "License to Drive."

Tough Stuff

17. Prove that the greatest common factor between $F(a)$ and $F(b)$ is also a Fibonacci number. But which one?

18. Find a two-term recurrence that has period 6: for any $n \geqslant 0$, $f(n+6) = f(n)$ and there is no smaller n for which this is true.

19. Find a two-term recurrence that has period 8.

20. What's this sum! $F(n)$ is the nth Fibonacci number.

$$\sum_{n=0}^{\infty} \frac{F(n)}{2^n} = \frac{0}{1} + \frac{1}{2} + \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots$$

Starting Pairs:

$B \begin{bmatrix} 4 \\ 14 \end{bmatrix}$ $E \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $T \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $H \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

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