Day 8: [You Spin Me Right]

Opener

1. Put three White balls and three Grey balls into a cup. You'll pull the balls randomly out of the cup one by one, keeping score with the number of White and Grey balls you've pulled. Grey wins the game by taking the lead at any time. White wins only if they do not trail *at all* while all six balls are pulled.

   For three minutes, run this game as many times as you can, tallying the number of wins for White and Grey.

It's a battle! White is good at chemistry, while Grey is good with ropes.

If the first ball is Grey, stop! Grey just won 1-0.

3

# of games in which White wins	# of games in which Grey wins	total # of games

2. This time, put four White balls and four Grey balls in your cup. The game rules are the same as before.
   a. Before you begin, make a prediction about whether White is *more* or *less* likely to win with 4 balls, compared to 3.
   b. For three minutes, run this game as many times as you can, tallying the number of wins for White and Grey.

White said something about being the one who knocks. Grey misinterpreted this as something altogether different.

Grey is Canadian, and said that the winner of this game should receive some sort of Grey Cup. That's pretty presumptive of a win . . .

4

# of games in which White wins	# of games in which Grey wins	total # of games

3. Repeat for five White balls and five Grey balls.

5

# of games in which White wins	# of games in which Grey wins	total # of games

Important Stuff

4. When your table has finished all three games, combine your data and pick someone to enter it here:

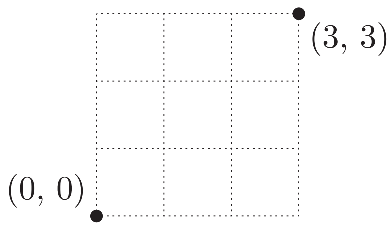
   `http://bit.ly/pcmi2017`

   Estimate the overall probability that White wins each game.

White says we'd better call Saul for this, while Grey put someone named Anastasia in charge of the business side of things.

5. Brooke wants to get from the origin to $(3,3)$ in the plane. She can only take single steps, one space to the right or one space up.

Different paths can intersect: "up up right right" is different from "up right up right". Sadly for Konami players, down and left are not allowed.



a. How many different ways can Brooke travel from $(0,0)$ to $(3,3)$?

b. How many different ways can Brooke travel from $(0,0)$ to $(4,4)$?

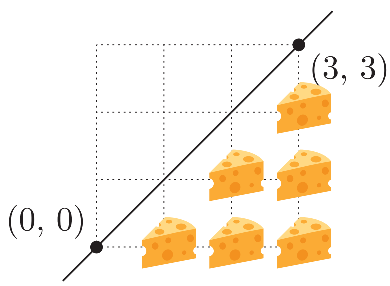
c. . . . from $(0,0)$ to $(5,5)$?

When Brooke travels to $(4,4)$, she doesn't have to go through $(3,3)$.

6. Kate also wants to get from the origin to $(3,3)$ in the plane. She takes single steps, one space to the right or one space up. Unlike Brooke, Kate is deathly afraid of cheese, which occupies all of the grid points strictly below the diagonal line connecting the origin to $(3,3)$.

There was an incident involving a cheesehead. Let's not ask.

She can be *on* the diagonal line, she just can't be *below* the diagonal line.



a. How many different ways can Kate travel from $(0,0)$ to $(3,3)$ that avoid the cheese?

b. . . . from $(0,0)$ to $(4,4)$?

c. . . . from $(0,0)$ to $(5,5)$?

What cheese is made backwards? . . . Edam. Man that pun was really . . . pretty awful.

7. Brooke has a list of all the ways she could travel from $(0,0)$ to $(3,3)$. Kate picks one of these ways at random.

a. What is the probability that Kate picks a route to $(3,3)$ that avoids the cheese? Use your previous work to help!

b. What changes if Brooke's routes go to $(4,4)$?

c. . . . to $(5,5)$?

If Kate picks successfully, the cheese will stand alone. Hi ho, the derry-o. And yes, that really is how to spell that line.

Neat Stuff

8. Here's a rule that maps points into new ones:

$$(x, y) \mapsto (y, -10x + 7y)$$

a. Find a point $(1, k)$ that gets doubled to $(2, 2k)$ by this rule.

b. Find some points (x, y) that get doubled by this rule. In other words, find (x, y) that map to $(2x, 2y)$.

c. Find some points (x, y) that get tripled by this rule.

d. Find some points (x, y) that get quintupled by this rule.

But but you said (x, y) maps to $(y, -10x + 7y)$. You're saying it could map to *both* that and $(2x, 2y)$ at the same . . . ohhh.

9. Here are the matrices from Day 4 showing the number of ways a person could travel between locations of the Hsu Shay Resort using one, two, or three tokens.

$$\begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 11 & 2 & 1 & 2 & 1 \\ 6 & 6 & 0 & 3 & 3 \\ 17 & 4 & 1 & 6 & 3 \\ 12 & 2 & 0 & 10 & 2 \\ 9 & 6 & 1 & 4 & 3 \end{bmatrix}$$

two-token grid one-token grid three-token grid

The fourth column of the three-token grid is the sum of the first and third columns of the two-token grid.

a. Using the resort map, explain why this happens.

b. Using the matrices, explain why this happens.

c. Look for some other relationships of this kind.

Hsu Shay is branching out into whip and boot manufacturing, and services for teenage love and stalking. Regardless, it's better than free fallin'.

10. **a.** List all the possible ways White can win the three-ball game from the Opener.

b. List all the possible ways White can win the four-ball game from the Opener.

c. Connect this work to the work from Day 7's coins and/or Crhyme schemes.

Assume White uses only legit, legal means to win, and that none of the balls have been crushed into a crystal form.

11. How many of Kate's paths to $(5, 5)$ go through $(4, 4)$? How many go through $(3, 3)$? Look for some possible ideas here to help describe a recursive rule for the total number of these paths.

Breaking up the problem into smaller ones, hmm. Interesting.

12. Here are the matrices from Day 4 showing the number of ways a person could travel between locations of the Hsu Shay Resort using one, two, or three tokens.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 0 & 3 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 9 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 11 & 2 & 1 & 2 & 1 \\ 6 & 6 & 0 & 3 & 3 \\ 17 & 4 & 1 & 6 & 3 \\ 12 & 2 & 0 & 10 & 2 \\ 9 & 6 & 1 & 4 & 3 \end{bmatrix}$$

one-token grid   two-token grid   three-token grid

Weekend trips to Hsu Shay are finally available for PCMI participants! Sign up on the board. Please no more transcendental or imaginary numbers of participants, unless your car is specifically built to handle these numbers.

The second row of the three-token grid is triple the first row of the two-token grid.

a. Using the resort map, explain why this happens.
b. Using the matrices, explain why this happens.
c. Look for some other relationships of this kind.

13. For each row of Pascal's Triangle, calculate its mean squared value.

For example, the *values* in one row are 1, 4, 6, 4, 1. The *mean squared value* is the mean of ... something.

14. For each recursion on points, find all the *scaled points*, points where (x, y) maps to a multiple of itself (kx, ky).

a. $(x, y) \mapsto (2x, 2y)$
b. $(x, y) \mapsto (x, 2y)$
c. $(x, y) \mapsto (-x, y)$
d. $(x, y) \mapsto (y, -21x+10y)$
e. $(x, y) \mapsto (y, 3x + 2y)$
f. $(x, y) \mapsto (y, x + y)$

Tough Stuff

15. Show how to divide Brooke's paths from $(0, 0)$ to (n, n) into $(n + 1)$ equal sets, exactly one of which is the set of Kate's paths.

16. Daniel, Daniel and Daniel give you a fun nonlinear recurrence:

$$\varepsilon(n) = \varepsilon(n-1) \cdot (2 - \ln \varepsilon(n-1)) \quad \text{with} \quad \varepsilon(0) = 1$$

What is $\varepsilon(1)$? What is $\varepsilon(5)$? What's going on?

Sure the recurrence is fun, but which Daniels are the fun Daniels? Can't they all be fun? No, they cannot.

17. Find a rule like the ones in problem 14 so that every point (x, y) maps to itself after exactly 5 iterations.

And no, $(x, y) \mapsto (x, y)$ is not allowed. This is Tough Stuff, fool.