Day 9: What Is The Point??

Opener

1. We're going to start with doing the same thing, over and over. Here's a rule that maps points into new ones:

Today's problem set is sponsored by Nathan Sykes. Also,

$$(x, y) \mapsto (4x + 2y, x + 3y)$$

a. Triangle EAT has points $E(0, 1)$, $A(-1, 1)$, $T(-1, 3)$. Determine the area of the triangle.

b. Transform triangle EAT according to the rule. What are the new coordinates and new area?

EAT again? We just had breakfast.

c. Transform a second time. What are the new coordinates? What's your guess about the new area?

d. What points (x, y) get scaled by a factor of 2 by this rule? There is more than one!

Such points would get mapped from (x, y) to $(2x, 2y)$. But we also said the rule maps (x, y) to $(4x + 2y, x + 3y)$. Oh!!

e. What points (x, y) get scaled by a factor of 3 by this rule? There isn't more than one!

f. What points (x, y) besides the origin get scaled by a factor of k by this rule? Find both k and the points that go along with those values of k. Within your table, have some people try to determine the values of k first, while others try to determine the points first.

Yes, you are trying to "solve" two equations involving three variables. Don't worry—it will all work out.

Important Stuff

2. **a.** The unit square has corners $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$. What's the area of the unit square?

b. Transform the unit square using the rule above. What shape is formed and what is its area?

3. This is your last chance. After this, there is no turning back. You take the blue pill: the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill: you stay in Wonderland, and I show you how deep the rabbit hole goes. Remember: all I'm offering is the truth. Nothing more.

Which pill will you EAT? Fortunately, the pills aren't triangular.

Day 9: What Is The Matrix??

Opener

1. We're going to start with doing the same thing, over and over. Here's a rule that maps points into new ones:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Today's problem set is sponsored by forks. There is no spoon, but there are forks!

a. Triangle ATE has points $A \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $T \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $E \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Determine the area of the triangle.

Good thing you ATE that red pill.

b. Transform triangle ATE according to the rule. What are the new coordinates and new area?

c. Transform a second time. What are the new coordinates? What's your guess about the new area?

d. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ get scaled by a factor of 2 by this rule? There is more than one!

Such points would get mapped from $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. But we also said the rule maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} 4x+2y \\ x+3y \end{bmatrix}$. Oh!!

e. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ get scaled by a factor of 3 by this rule? There isn't more than one!

f. What vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ besides $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ get scaled by a factor of k by this rule? Find both k and the points that go along with those values of k. Within your table, have some people try to determine the values of k first, while others try to determine the points first.

Yes, you are trying to "solve" two equations involving three variables. Don't worry—it will all work out.

Important Stuff

2. **a.** The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's the area of the unit square?

b. Transform the unit square using the rule above. What shape is formed and what is its area?

3. **a.** If $v(0) = 7$ and $v(n) = 5v(n-1)$, find a closed rule for $v(n)$.

b. If $v(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $v(n) = 5v(n-1)$, find a closed rule for $v(n)$ that looks like $v(n) = \boxed{}\, v(0)$.

c. If $v(0)$ is any vector and $v(n) = Av(n-1)$, find a closed rule for $v(n)$ that looks like $v(n) = \boxed{}\, v(0)$.

Phew, I was expecting another blank page then questions about triangle TEA.

A could be a number or it could be a matrix! But it can't be a matrice, because that doesn't exist.

4. Here's a rule that maps points into new ones:

$$v(n) = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} v(n-1)$$

Here, $v(n)$ is a vertical vector like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a. Transform the unit square using the rule above. What shape is formed and what is its area?

Whoa.

b. Starting with $v(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Find a vector to use for $v(0)$ so that $v(1)$ is a multiple of $v(0)$. Then compute $v(2)$ and $v(3)$ for that vector.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

Use a method that is different from the one that you used in problem 1f.

Is the matrix real? If "real" is what you can feel, smell, taste and see, then "real" is simply electrical signals interpreted by your brain. Your mind makes it real. No one can be told what the matrix is. You have to see it for yourself.

5. Here's another rule that maps points into new ones:

$$v(n) = \begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix} v(n-1)$$

Here, $v(n)$ is a vertical vector like $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a. Transform the unit square using the rule above. What shape is formed and what is its area?

Woah.

b. Starting with $v(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Find a vector to use for $v(0)$ so that $v(1)$ is a multiple of $v(0)$. Then compute $v(2)$ and $v(3)$ for that vector.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

e. Compute this matrix product for $n = 1, 2, 3$. What do you notice?

$$\begin{bmatrix} 1 & 1 \\ -5 & 7 \end{bmatrix}^n \begin{bmatrix} 0 & 1 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

Pretty sure the only Matrix product is batteries, right? Something like that. Hey, that movie is old enough to vote now.

Neat Stuff

6. Give an example of two matrices A and B such that the product AB can be computed but BA cannot.

Finally, proof that the Matrix is not all-powerful.

7. Ayesha multiplies two matrices and gets this:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Oops, the middle matrix is missing! What is it!

8. This mathematical expression is invalid even though it has two opening and two closing parentheses:

$$(4+5))1-2($$

Ignoring the mathematical stuff apart from the parentheses, there are only two ways that a valid mathematical expression can use two sets of parentheses:

$$()() \qquad \text{or} \qquad (())$$

That's right! Ignore the mathematical stuff! Free your mind!

a. How many valid ways can three sets of parentheses be used in a mathematical expression?

b. . . . four?

c. Based on your work over the last few days, make a conjecture for the number of valid ways to that five sets of parentheses can be used, and look for an explanation of why this connection can be made.

Conjectures that the world is a machine-created illusion will be resolved with kung-fu fighting, I guess.

9. **a.** David has a vector. After using the map in problem 4, he got $\begin{bmatrix} 10 \\ 6 \end{bmatrix}$. What's his vector?

b. Jennifer has a vector. After using the map in problem 4, she got $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What's her vector?

c. Jasper has a vector. After using the map in problem 4, he got $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's his vector?

d. Diana has a vector with variables! After using the map in problem 4, she got $\begin{bmatrix} a \\ b \end{bmatrix}$. What's her vector?

I'm trying to free your mind. But I can only show you the door. You're the one that has to walk through it.

Really, Diana? Your vector has variables?? What.

10. **a.** Find a 2-by-2 matrix A that takes any point in the plane and rotates it 90 degrees clockwise.

b. Compute A^2 and A^4.

c. Find a 2-by-2 matrix B that takes any point in the plane and rotates it exactly 60 degrees clockwise.

d. Find a 2-by-2 matrix C . . . 45 degrees clockwise.

Do not attempt to "find" the matrix. The Matrix is everywhere.

11. **a.** After expanding $a(a+b)(a+b+c)$ and combining like terms, how many different terms result?

b. Repeat for $a(a+b)(a+b+c)(a+b+c+d)$. Hmmm.

Please do not just write this bigger.

12. Did you know the numbers from problem 8 are lurking in even rows of Pascal's Triangle? Connect what you find to this expression from Day 8:

$$\frac{1}{n+1}\binom{2n}{n}$$

You can make a difference to find them!

Tough Stuff

13. **a.** Find a 2-by-2 matrix D that takes any point in the plane and rotates it exactly 15 degrees clockwise.

b. Find a 2-by-2 matrix E that takes any point in the plane and rotates it exactly 7.5 degrees clockwise.

14. For nonnegative integers n, calculate

$$\frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\,dx$$