Day 10: The Matrix, Reinputted

Opener

1. Here's a recursive rule: $v(n) = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} v(n-1)$

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

b. Starting with $v(0) = \begin{bmatrix} 20 \\ 40 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

c. Starting with $v(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ compute $v(1), v(2), v(3)$. Hmmmm.

d. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

e. Compute this matrix product for $n = 1, 2, 3$. Hmmmmm.

$$\begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}^n \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

f. Starting with $v(0) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, see if you can determine $v(3)$ with a very small amount of work.

Hopefully you won't react to this problem set the way that most people reacted to the second Matrix movie.

It's just another of those things that make you go hmmmm.

Thank you, C&C Music Factory, for all your contributions. Wait, were they in the Matrix movies? There was that one scene with the house party in Zion National Park or whatever that was, right? Anyway you should probably stop reading this because it has long since stopped being funny. No, really, this isn't going to get any better, and yet here you are, continuing to read this all the way to the end, while you could be solving the next problem. That really is a thing to make you go hmm.

Important Stuff

2. **a.** The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What's the area of the unit square again?

b. Transform the unit square using the rule

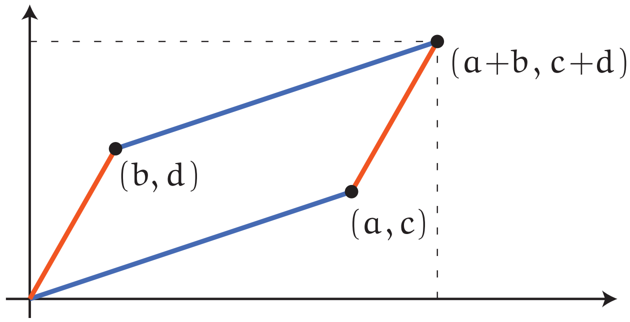
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 5 & 2 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What are the coordinates of the new shape, and what is its area?

Did you get the correct area? Perfect.

3. Find the area of this parallelogram. Do not continue to the next problem until you talk to someone else at your table who did this a different way.

Hey everybody, did you know that the d—*Andy, get out of here! Shh!*



4. John wants to know the scale factors for the rule

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 8 & -10 \\ 6 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

He doesn't need to know the vectors that get scaled by these scale factors. Help him find the scale factors with minimal algebraic impact.

Don't just make up numbers! John will know if you are lying.

Neat Stuff

5. **a.** Becky multiplies two matrices and gets this:

$$\begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix} \begin{bmatrix} \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix}$$

Oops, the middle matrix is missing! What is it!

The matrix seems to have no effect on it. It is . . . the one!

b. Benjamin multiplies two matrices and gets this:

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 41 & 42 \\ \pi & 0 \end{bmatrix}$$

Oops, the first matrix is missing! What is it!

Huh, this other matrix is also . . . the one!

Oops, we used "Oops" again.

6. **a.** Give an example of two matrices A and B such that the product AB can be computed but BA cannot.

b. Give an example of two matrices A and B such that both products can be computed but $AB \neq BA$.

c. Give an example of two distinct matrices A and B such that $AB = BA$.

This problem is way more fun if you say it like "mattresses". Some examples of products computed on mattresses are . . . goodnight, everybody!

7. Here's a fun rule: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

a. David has a vector. After using this map, he got $\begin{bmatrix} 23 \\ 36 \end{bmatrix}$. What's his vector?

b. Jennifer has a vector. After using this map, she got $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What's her vector?

I'm trying to free your mind. But I can only show you the door. You're the one that has to walk through it.

c. Jasper has a vector. After using this map, he got $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. What's his vector?

d. Diana has a vector with variables! After using this map, she got $\begin{bmatrix} a \\ b \end{bmatrix}$. What's her vector?

Really, Diana? Your vector has variables?? Seriously what even.

8. Here's a super fun rule: $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

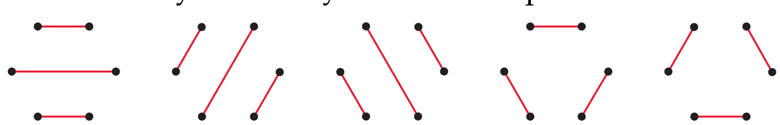
Apply this rule to $\begin{bmatrix} 23 \\ 36 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} a \\ b \end{bmatrix}$.

I'm beginning to doubt the accuracy of the fun level described in these problems.

9. What is the sum of the squares of the numbers in a row of Pascal's Triangle? Any thoughts about why this works?

10. Six people are seated at a round table. Three pairs of them simultaneously shake hands in such a way that their arms don't cross over each other. There are five different ways that they could accomplish this.

Assume everyone at the table has really long arms so that they can always reach across the table.



a. How many possible ways could 4 people simultaneously shake hands like this?

b. . . . 8 people?

c. Based on your work over the last few days, make a conjecture for the number of ways to perform these handshakes for 10 people or more.

11 people: 0 ways.

11. Redraw one of your handshake patterns for 8 people. Pick one of the vertices. (It doesn't matter which, but just be consistent every time you do this.) Write down a **U** because that vertex is connected to a new, unseen chord. Now go around the table clockwise. For each vertex, write down **U** (for unseen) if you meet a new chord you haven't seen before, or **R** (for rerun) if you meet a chord that you've already seen.

If you don't like using **U** and **R**, you can also use "**W**hat is this thing, I can't even..." and "**G**ot this boring thing already."

- **a.** Could this sequence of **U** and **R** ever have more **R** than **U** in it? Explain how you know.
- **b.** Use this to figure out how many possible handshake arrangements there are for $2n$ people.

It may help to do problem 8 from Day 9 if you haven't yet.

12. Hey, remember how you built that big matrix of numbers for the Crhyme schemes? Well . . .

Multiply these two matrices.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 5 & 5 & 3 & 1 & 0 & 0 \\ 14 & 14 & 9 & 4 & 1 & 0 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 & 1 & 0 \\ 0 & 0 & -1 & 6 & -5 & 1 \end{bmatrix}$$

Crhyme schemes often generate opportunities for the one. Remember that part of Matrix where there was that old guy in a chair, and they talked for like 10 minutes about weird boring stuff? Man, what was that. Anyway it was a Crhyme scheme that even made that scene possible. A heist!

Tough Stuff

13. What was that thing you just did in problem 12? Explain why it works.

14. How many combinations are possible on a 5-button Simplex lock? When you push a button on this lock, it stays in place and can't be pushed again. However, you can push more than one button at once, and activate a different combination. Not all 5 buttons need to be pushed in a combination, either. So . . . millions?

This must be one of them Fermi things. The Estimathon is tonight. It's not clear why they needed to know exactly how many people will be coming.

15. For nonnegative integers n, calculate

$$\frac{1}{2\pi}\int_0^4 x^n \sqrt{\frac{4-x}{x}}\,dx$$

16. We've been transforming simple shapes (like triangles and squares) with these maps. What happens to other shapes like circles?

It's Free Slurpee Palindrome Day! Go get a Slurpee!!