Day 11: The Matrix Rotations

Opener

1. Here's a recursive rule: $v(n) = \begin{bmatrix} 7 & -3 \\ 1 & 3 \end{bmatrix} v(n-1)$

a. Starting with $v(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ compute $v(1), v(2), v(3)$.

b. Solve this equation to determine the vectors that get scaled by this rule, along with their scale factors.

$$\begin{bmatrix} 7 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$$

c. Find α and β such that $\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

d. Use your answers to find a closed form expression for $v(n)$. Check to see if it agrees with your computation in part a.

So, if this is like the movies, today is the last day of the Matrix! Sadly the real world is not like the movies, especially not like the Matrix movies' version of the real world.

These letters are just fancy a and b so if you'd rather use that, do it. Or use emoji! Please do not use emoji as variables, especially not the Samsung cookie emoji. That is not a cookie. Cookie Monster gets super sad when he uses a Samsung phone for this reason.

Important Stuff

2. Here's another rule, in case you feel like we're doing the same thing, over and over again:

$$v(n) = \begin{bmatrix} 0 & 1 \\ -30 & 13 \end{bmatrix} v(n-1)$$

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$ compute $v(1), \ldots, v(4)$.

b. Determine the vectors that get scaled by this rule, along with their scale factors.

c. Write $\begin{bmatrix} 2 \\ 13 \end{bmatrix}$ as a combination of other useful vectors:

$$\begin{bmatrix} 2 \\ 13 \end{bmatrix} = \alpha \begin{bmatrix} \quad \\ \quad \end{bmatrix} + \beta \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

d. Use your answers to find a closed form expression for $v(n)$. Check to see if it agrees with your computation in part a.

You can do it! To celebrate, we're going to start calling these "I Can" vectors.

Someday you'll find it, the Matrix connection, the vectors, recursions, and you. All of us under its spell, we know that it's probably a malicious computer network . . .

3. For a 2-by-2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the *determinant* is $ad - bc$. This also seems to be the product of the two scale factors connected to a matrix.

Hey everybody, did you know the d—*Andy, we just talked about that!*

- **a.** Build a 2-by-2 matrix with determinant 12.
- **b.** If a matrix has determinant 12, what could its scale factors be?
- **c.** Investigate a way to take a 2-by-2 matrix and directly determine the *sum* of its scale factors.
- **d.** Build a 2-by-2 matrix with scale factors 6 and 3. Wow!

4. Here are Kayleigh's three favorite matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & -\frac{1}{2} \\ -2 & \frac{3}{2} \end{bmatrix}$$

- **a.** Calculate the determinants of A, B, and C.
- **b.** Calculate the products: AB, BA, AC, CA, BC, CB.
- **c.** Calculate the determinants of all the matrices from part b. What do you notice?

The Identity Mattress is so comfortable, you won't even know that it's there!

Nothing! I notice nothing! Good day, sir!

5. Matthew's favorite matrix is $D = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.

- **a.** Calculate the products AD, BD, CD, and the determinant of each. What happened?
- **b.** Draw the parallelogram represented by the columns of D. What happened?

Neat Stuff (feat. Matrices)

Choose Your Own Adventure! If you'd rather do some counting problems, go to page 46.

6. The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called the 2-by-2 *identity matrix*. What do you think it means for a matrix to be an *identity*? Compare to other uses of *identity* in mathematics.

Don't tell anybody! Then it can remain a secret identity.

7. Here's a recursive rule:
$$v(n) = \begin{bmatrix} -9 & 6 \\ 12 & 5 \end{bmatrix} v(n-1)$$

This matrix has scaled vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ with scale factor 9 and $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ with scale factor -13.

a. Starting with $v(0) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, compute $v(1), v(2), v(3)$.

b. Starting with $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$, compute $v(1), v(2), v(3)$.

Look for a way to do the first one with relatively little muss or fuss.

c. Write $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ as a combination of other useful vectors.

d. Starting with $v(0) = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$, find a closed form expression for $v(n)$.

These "I Can" vectors are really useful!

8. Here's a recursive rule: $$v(n) = \begin{bmatrix} 2 & 2 \\ 6 & 1 \end{bmatrix} v(n-1)$$

Starting with $v(0) = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$, say "I can do it!", then determine a closed form expression for $v(n)$.

This problem looks familiar, like deja vu. Maybe it's a glitch in the Matrix.

Whoa, you know kung $f(u)$.

9. Using the matrices from problem 4 . . .

a. Calculate the determinant of A^2.
b. Calculate the determinant of $3A$.
c. Calculate the determinant of $\frac{B}{10}$.
d. Calculate the determinant of $A + B$.

$3A$ means to multiply every number of the matrix A by the number 3.

10. **a.** Prove that when you multiply two 2-by-2 matrices, you also multiply their determinants.
b. If two matrices multiply together to make an identity matrix, what can you say about their determinants?

Boy, what *can't* you say about their determinants? They really are high-quality numbers. And definitely numbers, oh yes! No letters, that's for sure. These numbers are terrific.

11. **a.** Find a, c such that $$a\begin{bmatrix} 5 \\ 13 \end{bmatrix} + c\begin{bmatrix} 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b. Find b, d such that $$b\begin{bmatrix} 5 \\ 13 \end{bmatrix} + d\begin{bmatrix} 8 \\ 21 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

12. Find a, b, c, d such that

$$\begin{bmatrix} 5 & 8 \\ 13 & 21 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

They say you're supposed to invert your mattress every three months.

Neat Stuff (feat. Counting)

13. There are seven counting problems in this course so far that involve the *Catalan numbers*:

$$C(1) = 1$$
$$C(2) = 2$$
$$C(3) = 5$$
$$C(4) = 14$$
$$C(5) = 42$$
$$C(6) = 132$$

Look through previous problem sets to find them, and try the problems if you haven't had a chance.

The Opener for Day 7 is the first time the Catalan numbers appear. Problem 6 on Day 7 (Crhyme schemes) is the second appearance.

14. In each problem involving the Catalan numbers, you counted a certain set of objects. Since the number of these objects was the same each time, there should be some connection between the objects. Figure out some of the correspondences between these problems. Cool, this is problem 14.

There are 14 ways to stack coins with 4 coins on the bottom, and 14 Crhyme schemes for 4-line poems. So, there should be a way to turn each coin stack into a Crhyme scheme.

Once you find one connection, a new connection to either counts as a connection to both!

15. A *Simplex lock* has five buttons. A combination can involve any or all buttons, and more than one button can be pressed at a time. When a button is pressed, it locks in place and can't be pushed again, so pushing 3, then 4, then 3 is not possible. But pushing 2, then 3 and 4 at once, that's cool.

How many combinations are possible on a 5-button Simplex lock? Billions?

One combo is push 3, then 4, then 5, then enter the pool. Different: push 4, then 3, then 5. Different: push 3 and 4 at once and then 5. Different: push all of 3, 4, and 5 at once. Different . . .

Are there more combinations than square miles in Utah? Only Fermi knows for sure!

Tough Stuff

16. How many combinations are possible on a 6-button Simplex lock?

17. Establish a connection between the Simplex lock and some other set of numbers from this course.

18. The *Tribbiani sequence* is defined by the recurrence

$$f(n) = f(n-1) + f(n-2) + f(n-3)$$

If you start with any set of nonzero integers, the ratio of consecutive terms does . . . what?

The real Tribbiani sequence only has two terms: Friends, Joey.