Day 12: Eigen Do It!

Opener

1. Here's Kristy's favorite recursive rule: $v(n) = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} v(n-1)$

And Oyinka's favorite starting data: $v(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.

These two go great together! Time for a high west . . . I mean a high five.

a. Calculate $v(0), \ldots, v(3)$ using the rule, then complete this table. Refer to the Opener (part a) from Day 3 for the values of $J(n)$.

n	0	1	2	3
$v(n)$	$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$			
$J(n)$	2			

b. Determine a closed form expression for $v(n)$ using eigenvalues and eigenvectors. Check that your result gives the correct values for $v(1)$ through $v(3)$.

Use the 2-by-2 matrix to find those pesky "I Can" vectors! Through yesterday we called these *scale factors* and *scaled vectors*.

Important Stuff

2. **a.** The unit square has corners $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

What's the area of the unit square again?

Stop asking this question, over and over again!

b. Transform the unit square using Kate's rule

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \frac{7}{2} & -3 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What are the coordinates of the new shape, and what is its area?

c. Find the eigenvalues and eigenvectors of the matrix above.

d. Anna wonders what happens to the new shape if you repeatedly transform it using the matrix. So . . . ?

You say, she talks so all the time. So. A palindrome name for palindrome day!

3. Define $v(n) = \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ so that Kristy's rule becomes

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

Kristy's rule was mostly benevolent. Mostly.

a. Use matrix multiplication to build this system of equations:

$$\begin{aligned} x(n) &= y(n-1) \\ y(n) &= -10x(n-1) + 7y(n-1) \end{aligned}$$

Since $x(n) = y(n-1)$ for all n, that also means $x(n-1) = y(n-2)$ and $x(n+1) = y(n)$.

b. Take the two equations from part a and combine them into a single equation involving only x or only y. Connect that equation to the Day 3 Opener.

4. Melanie likes the recursive rule

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

with starting data $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

This rule is as easy as 0, 1, 2, 3! Okay fine it's 0, 1, 3, 2, whatever.

a. Say "I can do it!" and find a closed formula for $x(n)$.

b. Compare your work with the Monica sequence.

The Monica sequence was featured in the Opener on Day $C(2)$, where $C(n)$ is the Catalan numbers.

Neat Stuff (feat. Matrices)

5. **a.** Build a 2-by-2 matrix A so that the equation

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = A \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

models the recursion $x(n) = 2x(n-1) + 35x(n-2)$.

b. Find the eigenvalues and eigenvectors for matrix A.

6. Andrew likes this recursive rule:

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Andrew likes tea with his fractions. Tea, frac. Tea that frac! Tea, frac, tea that frac!

a. Calculate the eigenvalues and eigenvectors of the matrix above.

b. Abigail's favorite triangle has vertices $A(1,1)$, $B(1,3)$, $G(3,1)$. Graph this triangle.

You can do it! You've got this problem in the BAG.

c. Transform the triangle three times using Andrew's rule, graphing the new triangle each time. Notice anything?

d. Graph two lines through the origin that show all possible scaled vectors. Investigate how Abigail's triangle changes, relative to the graphed eigenvectors.

7. Try problem 10a from Day 9. Find eigenvalues and eigenvectors of your matrix. Don't be afraid of the complexity!

Just say "i can do it!"

8. Find the eigenvalues and associated eigenvectors for these matrices:

a. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ -6 & 6 & 4 \end{bmatrix}$ **b.** $\begin{bmatrix} 7 & 0 & 0 \\ 1 & 5 & 0 \\ 3 & 2 & 9 \end{bmatrix}$

9. If you pick two vectors from the origin, (a, c) and (b, d), the shape made from those vectors (and their sums) is a parallelogram. In three dimensions, the same concept determines a *parallelepiped* spanned by three vectors from the origin: (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

This could be the best math term ever. Parallelepiped! Say it three times: parallelepiped parallelepiped parallelepiped. Or perhaps Michael Keaton will come out of the box, or a parallelepiped parallelepiper might appear. A die made in this shape has parallelepips on it.

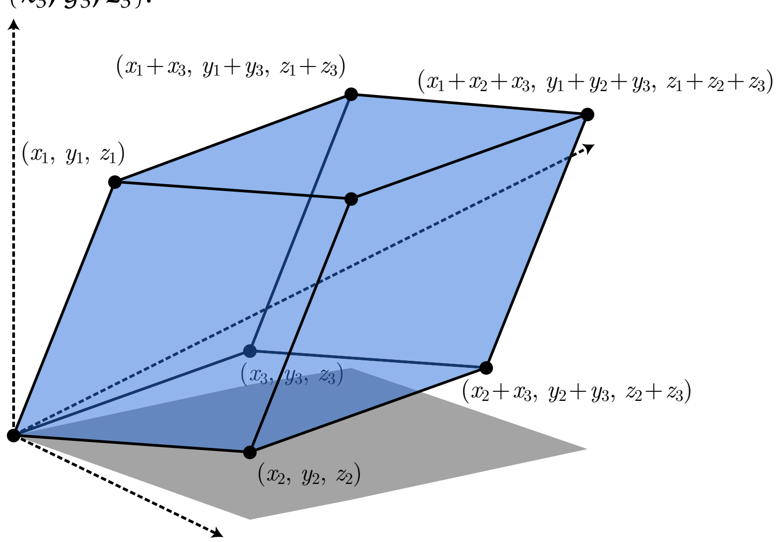

Figure adapted from `http://en.wikipedia.org/wiki/File:Determinant_parallelepiped.svg`.

Find the volume of the parallelepiped in terms of the nine variables. See if you can do this geometrically, without relying on a formula.

This problem is shear madness.

10. Convert this recursive rule involving a 3-by-3 matrix

$$\begin{bmatrix} x(n) \\ y(n) \\ z(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \\ z(n-1) \end{bmatrix}$$

into a three-term recursion rule for one variable. How might this work for four-term recursions?

11. Yesterday, Bowen made a claim that the final answer for the Opener would be the same if we had written the eigenvector as $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix}$. Test this by using different eigenvectors in Day 11's Opener.

Bowen also claimed to be in a Kentucky Fried Chicken commercial, and that Rosie O'Donnell called him cute.

12. Let $A = \begin{bmatrix} k & 1 \\ 0 & k \end{bmatrix}$.

a. Calculate A^2 and A^3. What will A^n be?

b. Consider the recursive rule

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x(n-1) \\ y(n-1) \end{bmatrix}$$

with starting data $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Instead of finding eigenvalues and eigenvectors, use your work from part a to find a closed form expression for $x(n)$ and $y(n)$.

13. Consider the recurrence relation

$$\begin{bmatrix} a(n) \\ b(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k^2 & 2k \end{bmatrix} \begin{bmatrix} a(n-1) \\ b(n-1) \end{bmatrix}$$

with starting data $\begin{bmatrix} a(0) \\ b(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2k \end{bmatrix}$.

Consider yourself, our matrix . . . we don't want to have no muss . . . for after some consideration we can say . . . consider yourself k^2x.

a. How does this relate to problem 18 on Day 5?

b. Make the substitution

$$a(n) = kx(n) + y(n)$$
$$b(n) = k^2x(n) + 2ky(n)$$

so that you no longer have a and b but instead have x and y.

c. Solve the new equations for $x(n)$ and $y(n)$ in terms of $x(n-1)$ and $y(n-1)$.

d. Use the previous problem to find closed form expressions for $x(n)$ and $y(n)$.

e. Reverse the substitution to obtain closed form expressions for $a(n)$ and $b(n)$.

f. Repeat this problem for arbitrary starting data $v(0) = \begin{bmatrix} p \\ q \end{bmatrix}$.

While you're visiting High West, be sure to ask them if you're allowed to order dinner. Chances are nonzero that they'll say no, even though you are seated at a table as a paying customer!

Neat Stuff (feat. Counting)

14. Sunny has flipped directly to the Counting problems, just like she did yesterday. She wants to know the number of different ways to tile a 2-by-10 rectangle with identical 1-by-2 dominoes. Consider any rotations or reflections to be different tilings.

And this is a *good thing*! Choose your own adventure!

15. Mary hates the number 1, and wants to count the number of ways to write 10 as a sum without ever using the number 1. For example:

$$\begin{aligned} 10 &= 8+2 \\ &= 2+8 \\ &= 3+4+3 \\ &= 4+3+3 \\ &= 3+3+4 \\ &= 10 \\ &= 2+2+2+2+2 \\ &= 5+5 \\ &= 2+6+2 \\ &= \cdots \end{aligned}$$

How many different ways can Mary write 10 like this?

Nobody tell her that there's a "1" inside the number 10, okay?

16. Becky built a chart to deal with the Simplex lock.

See Day 11 for a description of the Simplex lock. Okay, the zero-button lock and the zero-button combination are not very useful, you'll have to trust us on this one.

	# of distinct pushes					
	0	1	2	3	4	5
0-button lock	1					
1-button lock	1	1				
2-button lock	1		2			
3-button lock						
4-button lock		15				
5-button lock						

How many different combinations are there on a 5-button Simplex lock?

17. Use problem 10 from Problem Set 7 to establish this relationship among Catalan numbers:

$$C(n+1) = \sum_{i=0}^{n} C(i)C(n-i) \qquad \text{for } n \geqslant 0$$

with the starting data $C(0) = 1$. For instance, if $n = 3$,

$$C(4) = C(0)C(3) + C(1)C(2) + C(2)C(1) + C(3)C(0)$$

This problem is proof that Catalan numbers can march up and down, and definitely belong in a parade.

18. Use your work from Day 8 to establish this closed form expression for Catalan numbers:

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Tough Stuff

19. Of all the combinations on a 5-button Simplex lock, how many of the combinations involve pressing all five buttons, and how many involve less than five buttons? Hmmm! Looks like a proof might be useful here.

Hm. Is this problem simple, or is it complex? Or perhaps it is both. Simplex!

20. Find some recursive rules that are satisfied by $S(n)$, the total number of combinations on an n-button Simplex lock. Then find the number of combinations on a 10-button Simplex lock.