Day 13: Prime of My Life

Opener

1. Sean likes this recursive rule:

$$v(n) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} v(n-1)$$

a. Ariel's favorite triangle is $F(1,1)$, $L(2,1)$, $Y(2,3)$. Graph this triangle.

b. Transform the triangle five times using Sean's rule, graphing the new triangle each time. What do you notice about the triangle's location and coordinates?

c. Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

d. Find a closed form expression for $v(n)$ if

$$v(n) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} v(n-1)$$

with starting data $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Now I've had the prime of my life, no I've never factored this before. Yes I swear, I have proof, and I owe it all to you . . . clid.

Just like the closing dinner, this triangle seems to have unlimited rolls!

It's tricky to rock these eigs, to rock this problem right on time, it's tricky! How is it, D?

Important Stuff

2. Compute the area of each triangle in problem 1. Cool!

3. Gabe's favorite matrix is $\begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix}$.

a. Find eigenvalues and eigenvectors for Gabe's matrix.

b. Chris loves to cancel out Gabe! Find his matrix, the one that makes this equation true:

$$\begin{bmatrix} 1 & 4 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Find eigenvalues and eigenvectors for Chris's matrix. Oh, snap!

Sean says that if you stare at these triangles long enough, it'll show a 3-D magic eye pattern.

Careful, Gabe, or Chris will tell you the winner of The Genius Season 3 before you can watch it! Here's a hint: it's not Doohee.

4. Here's Gabie's recursive rule: $v(n) = \begin{bmatrix} -13 & 60 \\ -5 & 22 \end{bmatrix} v(n-1)$

   a. Find eigenvalues and eigenvectors of the matrix in her rule.

   b. Gabie is interested in the ratio between the top and bottom numbers in the vector $v(n)$, as n increases. She starts with $v(0) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Use what you've learned about eigenvalues and eigenvectors to predict what that ratio will be as n increases.

   c. Calculate $v(1), \ldots, v(5)$ by matrix multiplication (with or without techmology), and calculate the ratio of the top to bottom number in each vector. What happens?

Calculating for so long, now I finally found a 1 coprime to me . . . saw the writing on the wall, it's one of those vertical surfacies . . .

For example, $v(1) = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$, and the ratio is about 1.67.

Review Your Stuff

5. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 15 minutes** on this. Make sure your question is something that ∗**everyone**∗ at your table can do, and that you expect ∗**everyone**∗ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, the color of the paper on which you submit your question, your group's ability to write a good joke, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. Just say "I can do it!" Then actually do it.

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff (feat. Matrices)

6. Two lines are perpendicular at the origin. The sum of their slopes is 1. What are the equations of the lines?

7. Find eigenvalues and eigenvectors of these matrices.

   a. $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

   d. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Huh. That is worrisome...

You're the vector . . . I take multiples of. Now all this fancy math . . . fits like a glove! Because . . .

They try to make me act like other matrices, and I say no, no, no.

8. **a.** Find a, b, c, d to make this equation true:

$$\begin{bmatrix} 3 & 1 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity, and two matrices that multiply to an identity are inverses.

b. Find e, f, g, h to make this equation true:

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

c. Find a general formula for the inverse of a 2-by-2 matrix. In other words, find a matrix that makes this equation true:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

See problem 2 from Day 6, problem 5 from Day 10, problem 6 from Day 11.

The symbol A^{-1} denotes the inverse of a matrix A.

d. What must be true about a 2-by-2 matrix for it to have an inverse?

9. What must be true about a, b, c, d so that the system of equations

$$ax + by = e$$
$$cx + dy = f$$

has exactly one solution for x and y?

10. Andy tells you the formula for the determinant of a 3-by-3 matrix is

$$\det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_2y_1z_3 - x_1y_3z_2$$

Hey everybody, did you know the determinant of a 3-by-3 matrix is also the vol—*yes Andy, didn't you see the cool picture on Day 12?*

Calculate the determinant of the matrices in problem 8 from Day 12. Are the determinants related to the eigenvalues in any way?

11. Think about Gabie's experiment in problem 4. Find a starting vector, besides $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, that does *not* eventually lead to the same ratio of top to bottom numbers.

Every vector dance now! Every vector dance now! You better be ready, math nerds!

12. Here's another of those charts! This one gives the coefficients of expanded polynomials coming from $(x-1)(x-2)\cdots(x-k)$. Complete the chart and look for some recursions in it.

With imaginary i's there's no way we can disguise complexity . . . so we take each others' hand, trying to help us understand knot theory . . . just remember! You're the 1 thing, making identities . . . some of this matrix stuff, leads to obscenities!

	coefficient of power of x					
terms	x^0	x^1	x^2	x^3	x^4	x^5
1	1					
$x-1$	-1	1				
$(x-1)(x-2)$	2	-3				
3 terms		11				
4 terms	24				1	
5 terms						

The "3 terms" are $(x-1)(x-2)(x-3)$. Look, this one goes to 11!

13. Hm, both the chart in problem 12 and the chart of rhyme schemes from Day 6 are actually 6-by-6 matrices, if you let all the gray boxes be zeroes. I wonder what happens when they are multiplied.

14. You know, that means the chart of Crhyme schemes from Day 7, the one that generated Catalan numbers, is also a matrix that may have an inverse. (Oh it does.) Determine the inverse matrix and look up and down for some patterns in it.

If someone catches you inverting a matrix, just say it wasn't you. As long as you're not doing that on a bathroom floor, it should work.

15. Let's do one more chart! This one gives the coefficients of expanded polynomials coming from

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k)}{k!}$$

Complete the chart and look for some recursions in it.

Let's do this one more time. Oh oh oh. Can't stop, we're higher than a Mount Washington.

	coefficient of power of x					
	x^1	x^2	x^3	x^4	x^5	x^6
$\binom{x}{1}$	1					
$\binom{x}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$				
$\binom{x}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$				
$\binom{x}{4}$		$\frac{11}{24}$	$-\frac{1}{4}$			
$\binom{x}{5}$	$\frac{1}{5}$				$\frac{1}{120}$	
$\binom{x}{6}$						

Charts, charts, charts, charts charts charts! EVERYBODY!

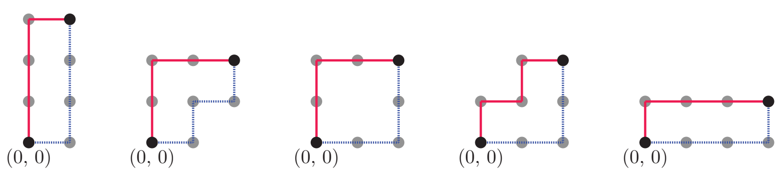
The Hsu Shay Resort would like to announce that all skiing "shoosh" sounds are being replaced by "shoop" sounds. Shoop Hsu Shay Shoobie, like Scoobie Doobie.

Dang, this matrix probably has an inverse, too. Alright, let's find the inverse.

Neat Stuff (feat. Counting)

16. Sam and Roy start at the origin in the plane. Sam moves up one unit and Roy moves to the right one unit. From that point on, each of them can either move up or right, but they don't run into each other again until they have moved exactly four spaces (including the first move). There are five different ways that they could do this.

Ohh, one two, hikers stand before you, that's what I said now, each one gonna take a different path now, just go ahead now. Neither has diamonds in his pockets, but they've got bread, now.



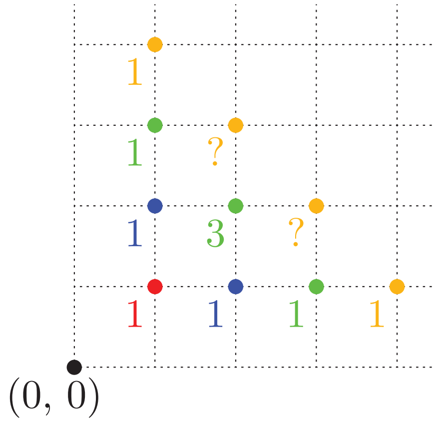
a. How many different ways could they do this if they each move exactly three spaces?

b. . . . five spaces?

c. Connect this problem back to work from previous problem sets.

17. Let's revisit Sam and Roy's problem from a different perspective. Given the constraints of how Sam and Roy are able to move, how many different ways can they meet up at each of the lattice points?

Ask Sean, maybe this is a magic eye 3-D question. That purple shirt, though.



a. This triangle of numbers connects to many of the problems we've seen (Crhyme schemes, Grey/White games, non-cheese paths, etc.). Find as many connections as you can.

b. Does this triangle of numbers build in a way reminiscent of Pascal's triangle? Figure stuff out!

Aw, this is going to be another one of those charts,isn't it.

The Sam-Roy dance, is your chance, to do some math. You know what we're doin', we're doing the Sam-Roy math.

Tough Stuff

18. Determine the value of this sum without the use of techmology.

$$\sum_{k=0}^{\infty} \frac{k^3}{k!} =$$

Extend to other powers of k.

19. Explain why the matrix inverse relationship between rhyme schemes and products of $(x-1)(x-2)(x-3)\dots$ exists.

I'm pretty sure it exists for the sake of problems like this one.

20. Find a 3-by-3 matrix A so that A^{12} is the identity matrix but no earlier power A^k with $k > 0$ is the identity. No cheating by rigging up a 2-by-2 matrix with an extra "1" in the corner.

That's 1 in the corner. That's 1 in the spotlight, losing a dimension.