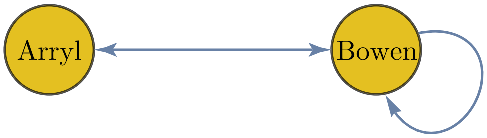
# Day 14: Crazy Ex Math Camp

---

## Opener

**1.** Given this ski resort:

This problem courtesy of Table 1!



**a.** Fill in this matrix representing the number of ways to travel between locations using exactly one token.

*... to*

*# ways from*

| | **A** | **B** |
|---|---|---|
| **A** | | |
| **B** | | |

**b.** Find the matrix representing the number of ways to travel between locations using exactly *ten* tokens. Bonus to those who can figure out how to do this without techmology!

**c.** If you start at B, what happens to the ratio of the number of ways to get to B to the number of ways to get to A as the number of tokens gets larger?

Hey everybody, did you notice that A and B from the Hsu Shay Resort were set up this—*oh, that's actually pretty interesting, Andy.*

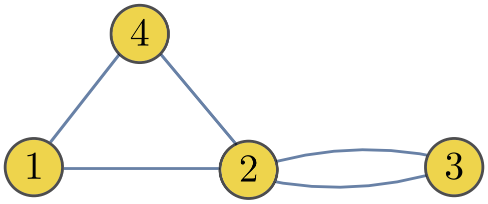
---

### Important Stuff

**2.** G is the graph ...

All paths go in both directions for this graph.

Oh the math of love triangles . . . is as simple as can be! Whichever Tom or Dick, I might pick, the center of the triangle is little ole me! (Actually, a triangle has multiple centers.)



**a.** Find the adjacency matrix A.

**b.** Find the matrix giving the number of three-step walks.

Do you like apples?

**3.** **a.** Vicki loves the matrix $A = \begin{bmatrix} 4 & 8 \\ 7 & 5 \end{bmatrix}$ and uses it in a recursive rule $v(n) = Av(n-1)$ with starting vector $v(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $v(1), v(2), v(3), v(4)$.

This is a terrible scandal, and we're sorry we had to reveal it in this way.

**b.** Will is impatient and wants to get bigger numbers faster so he defines $B = A^2$. Calculate B.

This will get him time with the baby faster.

**c.** Will uses B in a recursive rule $w(n) = Bw(n-1)$ with starting vector $w(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Calculate $w(1)$ and $w(2)$.

**d.** Calculate eigenvalues and eigenvectors for $A$ and B. Use these to write closed form expressions for $v(n)$ and $w(n)$. Notice anything?

**4.** Troy likes Monica's two-term recursive rule from Day 2

$$\begin{aligned} M(0) &= 2 \\ M(1) &= 2 \\ M(n) &= 2\,M(n-1) + 3\,M(n-2) \quad \text{if } n > 1 \end{aligned}$$

but he's impatient and wants to get bigger numbers faster. Write a new two-term recursive rule for $T(n)$ that skips every other term in Monica's sequence.

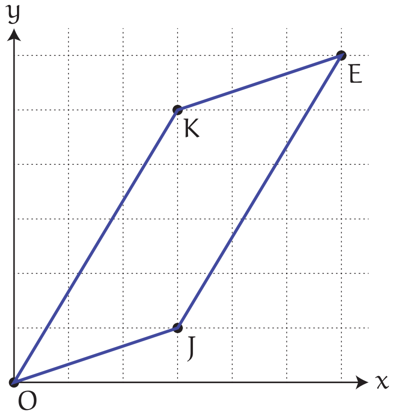
Troy's sequence skips over Monica's in this way: $T(0) = M(0)$, $T(1) = M(2)$, ..., $T(10) = M(20)$, etc. Skips, skips, skips, skips skips skips! EVERYBODY!

## Your Stuff

Your jokes. *Our jokes.*

**T2.** Find a matrix that takes the shape shown to the unit square.

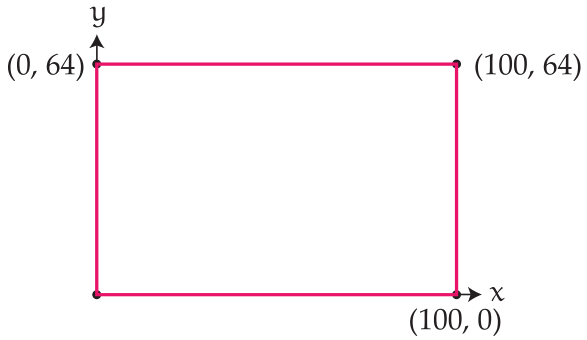
Look around for the joke.



*Oh, the math of love parallelograms . . . has a simple little trick. To find the area of the parallelogram . . . just use the theorem of Mr. Pick! (Actually that's only true for lattice points.)*

**T11.** Huey says "It's hip to be a rectangle," in a bid to gain more pop cultural relevance. He decides to transform all the unit squares in his condo to rectangles. One of this transformations looks like this:

*It's really more hip to be a triangle, now. Huey should run for governor of Louisiana in a bid to gain more literary relevance.*



**a.** Find a 2-by-2 matrix that will help Huey transform the unit square appropriately.

**b.** Huey isn't a "math person." Find two different matrices that can accomplish the same result with smaller, Huey-friendly entries.

*Huey really just has a very poor mindset. He believes in the power of love, but not powers of matrices.*

**T5.** **a.** Create a 2-by-2 matrix with eigenvalues 3 and 4.

**b.** Create a 2-by-2 matrix with one distinct real "I Can" value.

**c.** Does this matrix still create a parallelogram when starting with a unit square?

**T6.** Given $v(n) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} v(n-1)$ with $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

You know nothing, Jon Snow.

**a.** Find $v(1), v(2), v(3)$. Graph each.

**b.** Describe the transformation. Be specific.

**c.** How many mappings are necessary to get back to the original vector?

That's right! Going left is descriptive. Going left 12 units is specific.

**T7.** Sam's favorite triangle is $S \begin{bmatrix} 2 \\ 8 \end{bmatrix}, A \begin{bmatrix} 0 \\ 6 \end{bmatrix}, M \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

**a.** Find the area of $SAM$.

**b.** Sam's arch enemy triangle is $J \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E \begin{bmatrix} -2 \\ -2 \end{bmatrix}, N \begin{bmatrix} 6 \\ -7 \end{bmatrix}$.

Graph and find the area of $JEN$. Notice anything?

*I always thought the arch enemy was Burger King.*

**c.** Find the determinant of $\begin{bmatrix} -2 & 6 \\ -2 & -7 \end{bmatrix}$. What does this represent geometrically? What connections do you see?

**T8.** Melanie likes the recursive rule

$$v(n) = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix} v(n-1)$$

Imagine $i$, a matrix . . . You may say I'm a vector and I'm having fun. I hope some day you'll scale me, and this problem will be done.

**a.** Find the eigenvalues and eigenvectors for Melanie's matrix.

**b.** Melanie's favorite triangle is $I \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $D \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $K \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Graph IDK.

*Wait, I missed that. What triangle was it?* IDK.

**c.** Transform IDK five times using Melanie's rule, graphing the new triangle each time. What do you notice about the triangle's location and coordinates?

**d.** What happens if Melanie keeps applying her rule to IDK?

*All matrix work and no play make Melanie something something.*

**T4.** The inverse of matrix $A$ is $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Without directly finding $A$, write a closed form expression for $v(n) = A^n \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

**T9.** **a.** Solve for $k$: $0 = k^2 - 8k + 15$

**b.** Solve for $k$: $0 = 15k^2 - 8k + 1$

**c.** Take the quadratic from part b, substitute in $1/m$ for $k$ and solve for $m$. What do you notice?

**d.** Given MAD GT's favorite quadratics

$$0 = ax^2 + bx + c$$
$$0 = cx^2 + bx + a$$

what can you say about the solutions?

MAD GT is the name of Table 9's biker gang. Find them at the Cabin on Thursday nights. Don't get them confused with Happy GTs . . . they don't like being confused with the Happy GTs.

**T12.** Here's a three-term recursive rule:

$$F(0) = 0, F(1) = 1, F(2) = 1$$
$$F(n) = F(n-1) + F(n-2) + F(n-3) \quad \text{if } n \geqslant 3$$

**a.** Find the first 10 terms.
**b.** What patterns do you notice?
**c.** If you were going to find a closed form, what types of equations might you need to work with/solve?
**d.** Explore as desired.

Go cube-ic!

**T3.** Given a recursive rule

$$v(n) = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} v(n-1)$$

with starting data $v(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find the closed rule

$$v(n) = a\left( \quad \right)^n \begin{bmatrix} \\ \\ \end{bmatrix} + b\left( \quad \right)^n \begin{bmatrix} \\ \\ \end{bmatrix}$$

What is the general rule for the magic farm animals matrix $\begin{bmatrix} 0 & 1 \\ \text{cow} & \text{duck} \end{bmatrix}$?

**T10.** **a.** A volleyball team has 1 setter, 2 attackers, and 3 diggers. A sequence of hits must follow the following rules to win a point.

- The ball must be hit in this order: digger $\Longrightarrow$ setter $\Longrightarrow$ attacker.

Assuming players are equally likely to hit the ball to anyone, including themselves, find the probability of winning a point given the necessary sequence.

Hey everybody, I thought I told you no more volleyb—*let it go, Andy, let them play!*

**b.** Now we're playing a new Top Gun version of volleyball, which has the following rules.

- If a person hits a ball twice in a row, they instantly lose.
- You can, but you don't *have* to, have four hits on one side.
- To win a point, some set of three hits in a row must go digger $\Longrightarrow$ setter $\Longrightarrow$ attacker.

Now what's the probability of winning?

Also rules-shirts off, second hit must go to Goose.

No more Goose—he was ejected.

*To actually do this as a math problem you will have to ignore the previous two notes. Also Top Gun II is real.*

**c.** For Top Gun II (coming in 2019) you can have five hits on one side. What's the probability of winning?

**T4.** BØW3N is the hottest new AI developed by the Skynet Working Group specializing in dropping mad random rhymez. What are the odds that its randomly generated

10-line poem matches the rhyme scheme of this no-longer-copyright-infringing masterpiece?

*Just a small town girl*
*Livin' in a Prospector*
*She took the midnight bus goin' to Main Street*
*A Park City boy*
*Raised here nowhere near Detroit*
*He took the midnight bus goin' to Main Street*
*A cowboy on a smoky stage*
*No one even knows his age*
*For a dollar they can share a song*
*That goes on, and on, and on, and on*

Oh, the movie never ends but we're from the days of VHS so you can rewind to Day 6.

Bonus: make a karaoke-worthy randomly generated song. *Done. Oh, whoops, we changed yours.*

### Neat Stuff (feat. Matrices)

**5.** Calculate eigenvalues and eigenvectors for these matrices. Oh, and their determinants too.

So, you want me to determine it?

**a.** $\begin{bmatrix} -1 & -6 & -3 \\ -9 & -4 & -8 \\ -9 & 7 & -4 \end{bmatrix}$ **b.** $\begin{bmatrix} -4 & 4 & -5 \\ -4 & 1 & -5 \\ -2 & -4 & -1 \end{bmatrix}$ **c.** $\begin{bmatrix} -1 & 1 & -2 \\ 3 & -6 & -3 \\ 2 & 1 & -5 \end{bmatrix}$

**6.** The *diagonal* of a square matrix goes from top left to bottom right. The *trace* of a square matrix is the sum of numbers on its diagonal.

The other diagonal is not a diagonal! Deal with it!

- **a.** How is the trace useful when finding eigenvalues of 2-by-2 matrices?
- **b.** Calculate the trace of each of the matrices above. Calculate the sum of the eigenvalues for each of the matrices above.

**7.** What happens to the determinant of a matrix when you perform these operations below? Make up some sample matrices to play with or use the ones above.

- **a.** Swap any two rows in a square matrix.
- **b.** Swap any two columns in a square matrix.
- **c.** Multiply an entire row or column by 10.
- **d.** Multiply the entire matrix by 10.

Hey everybody, did you know the determinant is a function on the TI—*it's a function on all the TI's, Andy!*

**e.** Replace the first row of the matrix by the sum of the original first and second rows.

**f.** Replace the first row of the matrix by the sum of the original first row and 10 times the second row.

**8.** In problem 3 on Day 13 you noticed the behavior of eigenvalues and eigenvectors for an inverse matrix.

**a.** Describe why you think this might always happen.

**b.** Under what circumstances will a matrix *not* have an inverse?

**9.** Multiply these. What?

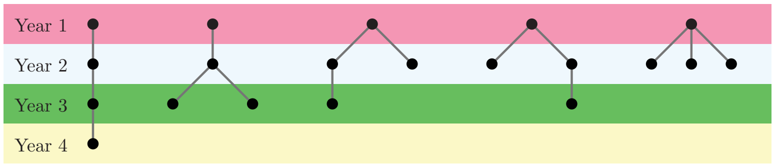
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 & 0 \\ 0 & 0 & 6 & 10 & 1 & 0 \\ 0 & 0 & 1 & 20 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 3 & -3 & 1 & 0 & 0 & 0 \\ -17 & 17 & -6 & 1 & 0 & 0 \\ 152 & -152 & 54 & -10 & 1 & 0 \\ -1943 & 1943 & -691 & 130 & -15 & 1 \end{bmatrix}$$

Oh, the math of love mattresses . . . is super super fun! Just multiply on two mattresses . . . and you will finally find the one! (Actually that only works for inverse matrices.)

## Neat Stuff (feat. Counting)

**10.** In Year 1 of PCMI, the Awesome Lessons Working Group made an Awesome Lesson that was so awesome that in later years, the group didn't make new lessons. Instead they chose to tweak the previous' year's versions into any number of variations, or just stop.

But you didn't have to come to Utah
Meet some friends and do recursion and learn 'bout eigenvalues
I guess you've got to leave us though
Now we're just some math camp that you used to know



Suzanne found that the working group created four lessons, including the original Awesome Lesson. There are exactly five ways in which the original Awesome Lesson could have evolved, shown above.

**a.** Suzanne found a different Awesome Lesson chain with 3 lessons. How many different ways could this Awesome Lesson have evolved?

**b.** Suzanne found another different Awesome Lesson chain with 5 lessons. How many different ways could this Awesome Lesson have evolved? Break down this total into categories, given that the working group could have been doing this for 1, 2, 3, 4, or 5 years.

**c.** Catalan this sucker. Relate this problem to others.

**11.** Mary is back, and doesn't hate the number 1 like she did on Day 12. But now she hates all even numbers! She wants to count the number of ways to write numbers as sums without ever using an even number. For example:

$$\begin{aligned} 10 &= 7+3 \\ &= 3+7 \\ &= 7+1+1+1 \\ &= \cdots \end{aligned}$$

How many different ways can Mary write the number $n$ like this?

Good thing for her this is problem 11. Or problem 23. Either way, she's good.

### Tough Stuff

**12.** How many ways are there to tile a 2-by-$n$ rectangle with rectangles of integer side lengths?

**13.** What percentage of the time will a Fibonacci number $F(n)$ and its corresponding Catalan number $C(n)$ share a common factor greater than 1?

**14.** Each of the entries of an $n$-by-$n$ matrix $A$ are randomly chosen (with equal probability) to be a 0 or 1. What is the probability that $A$ has an inverse? What happens to that probability as $n$ goes to infinity?

Now and then I think of all the times you gave me Neat Stuff
But had me believing it was always something I could do
Yeah I wanna live that way
Solving problems on a Saturday
But now you've got to let us go
And we're leaving from a math camp that you used to know

### No More Stuff

**15.** Thanks. We had a wonderful time and hope you did too. See you again as soon as possible.

We miss you so, so bad. See you on the next exciting episode of *The Gone Show*! Oh wait, we're canceled.