Day 1: Quack Each Duck

Welcome to PCMI! We hope you'll learn a great deal of mathematics here—maybe some new ideas, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.

PCMI participants have solved at least two previously unsolved problems presented in these courses.

- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about the Magic Box, think again: the problems should lead to the appropriate mathematics rather than requiring it.

What? Is the Magic Box a real math thing? Only Brad Pitt knows for sure.

- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That's* why it's Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

When you get to Day 3, come back and read this again.

Will you remember? We'll find out . . .

Opener

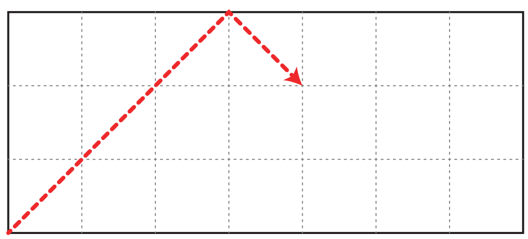
1. **a.** What fraction with denominator 10 or less is closest to π?
   b. What fraction with denominator 100 or less is closest to π? Make a plan at your table to divvy up the work.

Errr... You can't use $\pi/1$. You're only allowed whole numbers for numerator and denominator of your fraction.

Important Stuff

2. Today is Evie's birthday and she got a laser! It just so happens that she lives in a 7-by-3 rectangular room with mirrors for walls. So of course the first thing Evie does is fire a laser beam at a 45° angle from the lower-left corner of the room to see how the laser beam bounces off the walls.

Ours is not to wonder why, unless we're wondering how Mexico is playing like this.



Eventually, the laser path reaches a corner of the room. How long is Evie's entire laser beam path until it reaches a corner?

Experiment with different room sizes. Use the collective brain power of your table to gather information about many different room sizes until you can predict the length of the laser beam path for a m-by-n room, where m and n are positive integers.

You get grid paper! And YOU get grid paper! Grid paper for everyone!

Hopefully the m and n you pick don't make you feel like you're in that episode of The Office with the screensaver.

3. **a.** Aziz says that Evie's 7-by-3 room is too small. He wants to see the laser fired in a 28-by-12 room. What changes? What stays the same?
   b. Carol would prefer to see the laser fired in a 12-by-28 room. What changes? What stays the same?

4. Linda loves trig, so she picks her two favorite positive integers: 30 and 45.

The *greatest common divisor* of 30 and 45 is 15.
The *least common multiple* of 30 and 45 is 90.

15 and 90? Linda loves those too!

a. Find the GCD and LCM for 21 and 6; for 28 and 12.
b. Find the GCD and LCM for some other pairs.
c. Describe in your own words how to calculate the GCD and LCM of two numbers.
d. Something interesting happens to the sum or the product of GCD and LCM. One of those.

The *least common denominator* is a specific example where the LCM is useful. Have you heard the new Least Common Denominator Soundsystem album?

5. Find the fraction with the smallest possible denominator that's less than $\frac{1}{100}$ away from $\frac{1}{2}$, but isn't exactly $\frac{1}{2}$.

I don't understand what this problem has to do with the Groundwater Conservation District or Local and Connectional Ministries.

Neat Stuff

6. Which corner of the room did Evie's laser beam reach in her original 7-by-3 room? Experiment with different room sizes. Predict how the size of the room affects where the laser beam ends up.

Hey, why don't you and the laser beam get a room?! Oh, right.

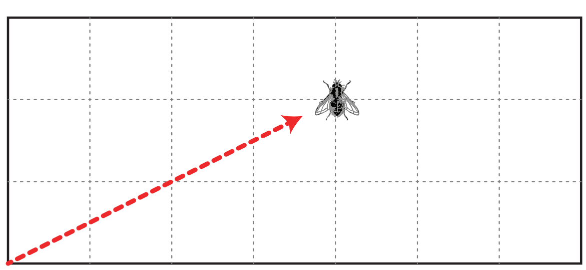
7. Evie's room has a floor tiled with 1-by-1 squares, as shown above. How many of these squares did Evie's laser beam cross over? Experiment with different room sizes. Predict how the size of the room affects the number of squares that are crossed.

8. **a.** Find a fraction that you would consider to be close to $\sqrt{5}$.
b. Find a fraction that is even closer to $\sqrt{5}$.
c. Find a fraction that is yet even closer to $\sqrt{5}$.

Maybe they're just friends?

9. How many times does Evie's laser bounce off walls in the 7-by-3 room before hitting a corner? Find a general rule for the total number of bounces in an m-by-n room.

10. Jane notices there is a bug hanging out in Mary's 7-by-3 room, so she tries to hit it with Evie's laser. She misses but the laser beam continues along that path. Figure out stuff about the path of the laser.



The laser is pointed at $(4, 2)$. Assume, unlikely as it may seem, that there are no other bugs in Mary's room.

11. **a.** Find the smallest positive integer n so that $\frac{1}{n}$ is less than the value of $|\frac{22}{7} - \pi|$.
b. What could this information be useful for?

12. Suppose a and b are positive integers with no common factor greater than 1. In how many places will a straight path from $(0,0)$ with slope $\frac{b}{a}$ cross a lattice line before hitting the point (a,b)?

$x = 3$ and $y = 4$ are lattice lines. Golden Corral has a lettuce line.

13. If a and b are positive integers with GCD d and LCM m, prove that

$$a \cdot b = d \cdot m$$

14. When Sweden plays Denmark, the scoreboard displays SWE-DEN. And, the letters that are unused spell DEN-MARK. Find other team names for which this is true.

Locals loved the last Super Bowl between New England and Philadelphia.

Tough Stuff

15. Betty's room is 3-dimensional and measures 3-by-7-by-11, with mirrors. She fires a laser beam from the lower left corner $(0,0,0)$ toward a speck of dust at $(1,1,1)$.
a. What corner of the room does the laser beam hit? Generalize to other sizes.
b. How long is the path of the laser beam? Generalize!

When the freakin' laser beam hits an edge that isn't a corner, it rebounds in two dimensions but continues the same direction in the third.

16. Like problem 9, determine the total number of bounces in a three-dimensional, L-by-W-by-H room.

17. Suppose a and b are irrational numbers so that

$$\frac{1}{a} + \frac{1}{b} = 1$$

Make a list of the multiples of a and b as decimals. Notice anything? Try to prove what you find.

Try picking a value for a, perhaps $\sqrt{2}$, to test the concept.