Day 13: There's Leggy Penguins?

Opener

1. **a.** Build the continued fraction for $\sqrt{14}$ and its first five convergents.
   b. For each convergent $\frac{y}{x}$, calculate $y^2 - 14x^2$.
   c. Find two solutions to $y^2 - 14x^2 = 1$ with $x, y > 0$.

The first two convergents are $\frac{3}{1}$ and $\frac{4}{1}$.

2. Calculate each of these.

   a. $(4+\sqrt{14})(4-\sqrt{14})$
   b. $(4+\sqrt{14})^2$
   c. $(30+8\sqrt{14})(30-8\sqrt{14})$
   d. $(15+4\sqrt{14})(15-4\sqrt{14})$
   e. $(15+4\sqrt{14})^2$
   f. $(449+120\sqrt{14})(449-120\sqrt{14})$

Important Stuff

3. **a.** Multiply this: $(y + x\sqrt{n})(y - x\sqrt{n}) =$
   b. If that product equals 1, and x and y are positive integers, what can you say about $(y - x\sqrt{n})$?

What's that product equal? No, it's Sweet'N Low.

4. **a.** Elli performs the Euclidean Algorithm (square cutting) to calculate the GCD of 477 and 152.
   b. Mark calculates the continued fraction for $\frac{477}{152}$ in the style of `bit.ly/pcmisqrt7`.
   c. Describe as many connections as you can.

5. Use Matthew's measure (see Problem 12 from Day 12) to determine which rational approximations you've encountered for π are the best.

They're the best . . . around! Nothing's ever gonna keep them down. Never gonna give them up.

6. Instead of measuring the difference between a fraction and a target number, Samantha measures the difference between the *square* of the fraction and the *square* of the target number. Use this idea to think about why integer solutions to $y^2 - nx^2 = 1$ produce such accurate rational approximations of $\sqrt{n}$.

Approximations to $\sqrt{n}$ have the right angle, but approximations to π are truly the best a-round.

Old Stuff

7. In the third century BCE, Archimedes posed a mathematics problem in a letter to Eratosthenes that in-

volved finding the number of white, black, dappled, and brown bulls and cows subject to several arithmetic conditions. (An English translation of that original problem by Hillion and Lenstra is attached.) Revel in the fact that this old problem amounts to finding positive, integer solutions to $y^2 - 410286423278424x^2 = 1$.

We'll be around to check your reveling.

8. Archimedes stated that

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

Do better than Archimedes.

Archimedes was good at math, but we're not sure why he cared so much about dappled things.

9. In the seventh century, Brahmagupta wrote this about the equation $y^2 - 92x^2 = 1$: "A person solving this problem within a year is a mathematician." Patricia challenges you to become a mathematician by finding two solutions to this equation for $x, y > 0$.

10. The Indian mathematician Ramanujan came up with this approximation

$$\pi \approx \left(\frac{2143}{22}\right)^{1/4}$$

which is even more accurate than 355/113, though it requires a root. Calculate the continued fraction for π^4 to reveal how he came up with this approximation.

When Ramanujan had trouble sleeping, he famously counted taxis instead of sheep, and usually didn't fall asleep until counting at least 1729 of them.

Review Your Stuff

11. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **one** review question for tomorrow's problem set. Spend **at most 20 minutes** on this. Make sure your question is something that $\boxed{\ast\textbf{everyone}\ast}$ at your table can do, and that you expect $\boxed{\ast\textbf{everyone}\ast}$ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, whether you've written your question on the approved piece of paper, your group's ability to write a good joke, how good your bribes are, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. You got this!

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff

12. A *Diophantine equation* is a polynomial equation, usually with two or more unknowns, for which integer solutions are desired. For example, Pythagorean triples are solutions to the Diophantine equation $a^2 + b^2 = c^2$.

a. Find an integer solution to $477x + 152y = 1$.
b. Find integer solutions to $y^2 - 11x^2 = 1$.
c. Use techmology to graph $y^3 - 5x^2 = 1$ and look for integer solutions with $x > 0$.
d. Use techmology to graph $y^2 - 5x^3 = 1$ and look for integer solutions with $x > 0$.

Need a hint? Use your work from Problem 4a.

13. You've now found many solutions to equations of the form $y^2 - nx^2 = 1$. Each solution (x, y) matches a fraction $\frac{y}{x}$ that serves as an excellent approximation to $\sqrt{n}$. But how excellent?

These approximations are most excellent, Ted! Way.

a. $\frac{15}{4} - \sqrt{14} \approx \frac{1}{M}$ for what closest possible integer M?
b. $\frac{9}{4} - \sqrt{5} \approx \frac{1}{M}$ for what closest possible integer M?
c. $\frac{1249}{200} - \sqrt{39} \approx \frac{1}{M}$ for what closest possible integer M?
d. Look for a pattern; if you need to, throw some more fractions at it, but make sure they are solutions to $y^2 - nx^2 = 1$.

14. Use Problems 3 and 13 to show that if (x, y) is a solution to $y^2 - nx^2 = 1$, then $\frac{y}{x} - \sqrt{n}$ is very, very close to $\frac{1}{2xy}$.

Start by exploring the approximate value of $(y - x\sqrt{n})$ in terms of x or y.

15. Revisit Problems 13 and 14 using other convergents, the ones that don't make the magical 1. What changes, what stays the same?

16. Fun with algebra! Rewrite the expression

$$(y_1^2 - nx_1^2)(y_2^2 - nx_2^2)$$

in the form

a. $(y_1y_2 + nx_1x_1)^2 - n(\qquad\qquad\qquad)^2$
b. $(y_1y_2 - nx_1x_1)^2 - n(\qquad\qquad\qquad)^2$

Fun with trivia! The director of *Happy Feet* and *Happy Feet 2* went on to direct his next movie . . . *Mad Max: Fury Road*. He also directed *all* the other Mad Max movies. He also wrote *Babe*, yes that one.

17. Calculators use this recursive formula to generate a decimal approximation for $\sqrt{n}$:

$$x_{m+1} = \frac{1}{2}\left(x_m + \frac{n}{x_m}\right)$$

One reason this works is because x_m and $\frac{n}{x_m}$ are on opposite sides of $\sqrt{n}$, but there are others.

[waves hands frantically, redirecting to Problem 18]

a. Roxanne uses the recursive formula with $n = 2$ and starting value $x_0 = 1$. What happens? Compare to your Box for $\sqrt{2}$.

b. Try $n = 7$ and $x_0 = 3$. What happens? Compare to your box for $\sqrt{7}$.

c. Try other values for n and x_0.

18. Ramona loves the Calculus! She applies Newton's Method to the equation $f(x) = x^2 - n$ to see where the formula in the previous problem comes from.

Newton's Method does not usually involve sitting under a tree and hoping to get hit by stuff.

19. Find some integer solutions for $y^2 - 5x^2 = -1$. Refer back to your Box that you previously built for $\sqrt{5}$, or build one.

20. Have you noticed that small coefficients seem more frequent than large ones in continued fractions? It turns out that for almost every real number, if a_k are the coefficients of its continued fraction, then there is a universal limit to the geometric mean for these coefficients:

$$(a_0 a_1 a_2 \cdots a_{n-1})^{1/n} \to 2.68545... \qquad \text{as } n \to \infty$$

That number is called Khinchin's constant. Try calculating the geometric mean of the first 5, 10, then 15 coefficients of the continued fraction for $\pi + e$. Experiment with other numbers.

This result doesn't work well for irrational solutions to quadratic equations with rational coefficients, because those continued fractions always repeat. But if you pick a positive real number at random, the chance that you'll get one of those is zero.

Tough Stuff

21. For which integers n does the equation $y^2 - nx^2 = -1$ admit integer solutions?

22. The denominators of rational convergents seem to get larger quickly, but they don't grow exponentially. Instead, prove that for almost every continued fraction expansion, the denominators of the convergents, x_n, grow like

$$x_n^{1/n} \to \exp\left(\frac{\pi^2}{12 \ln 2}\right) \qquad \text{as } n \to \infty$$

That limiting value is called Lévy's constant.

And the title of Leggiest Penguin Ever goes to . . . Danny DeVito? Yikes.