Day 1: Quack Each Duck

Welcome to PCMI! We hope you'll learn a great deal of mathematics here—maybe some new ideas, maybe some new perspectives on things with which you're already familiar. A few things you should know about how the class is organized:

- **Don't worry about answering all the questions.** If you're answering every question, we haven't written the problem sets correctly.
- **Don't worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you've encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don't be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don't mean you should ignore your classmates but give everyone the chance to discover. If you think it's a good time to teach your colleagues about the Magic Box, think again: the problems should lead to the appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That's* why it's Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

PCMI participants have solved at least two previously unsolved problems presented in these courses.

What? Is the Magic Box a real math thing? Only Brad Pitt knows for sure.

When you get to Day 3, come back and read this again.

Will you remember? We'll find out . . .

Opener

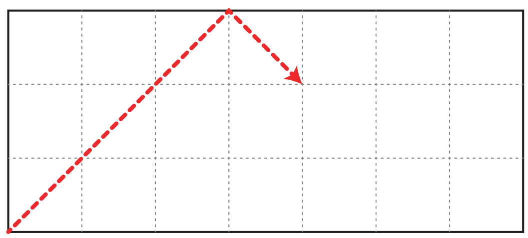
1. **a.** What fraction with denominator 10 or less is closest to π?
   b. What fraction with denominator 100 or less is closest to π? Make a plan at your table to divvy up the work.

Errr... You can't use $\pi/1$. You're only allowed whole numbers for numerator and denominator of your fraction.

Important Stuff

2. Today is Evie's birthday and she got a laser! It just so happens that she lives in a 7-by-3 rectangular room with mirrors for walls. So of course the first thing Evie does is fire a laser beam at a 45° angle from the lower-left corner of the room to see how the laser beam bounces off the walls.

Ours is not to wonder why, unless we're wondering how Mexico is playing like this.



Eventually, the laser path reaches a corner of the room. How long is Evie's entire laser beam path until it reaches a corner?

Experiment with different room sizes. Use the collective brain power of your table to gather information about many different room sizes until you can predict the length of the laser beam path for a m-by-n room, where m and n are positive integers.

You get grid paper! And YOU get grid paper! Grid paper for everyone!

Hopefully the m and n you pick don't make you feel like you're in that episode of The Office with the screensaver.

3. **a.** Aziz says that Evie's 7-by-3 room is too small. He wants to see the laser fired in a 28-by-12 room. What changes? What stays the same?
   b. Carol would prefer to see the laser fired in a 12-by-28 room. What changes? What stays the same?

4. Linda loves trig, so she picks her two favorite positive integers: 30 and 45.

   The *greatest common divisor* of 30 and 45 is 15.
   The *least common multiple* of 30 and 45 is 90.

15 and 90? Linda loves those too!

a. Find the GCD and LCM for 21 and 6; for 28 and 12.
b. Find the GCD and LCM for some other pairs.
c. Describe in your own words how to calculate the GCD and LCM of two numbers.
d. Something interesting happens to the sum or the product of GCD and LCM. One of those.

The *least common denominator* is a specific example where the LCM is useful. Have you heard the new Least Common Denominator Soundsystem album?

5. Find the fraction with the smallest possible denominator that's less than $\frac{1}{100}$ away from $\frac{1}{2}$, but isn't exactly $\frac{1}{2}$.

I don't understand what this problem has to do with the Groundwater Conservation District or Local and Connectional Ministries.

Neat Stuff

6. Which corner of the room did Evie's laser beam reach in her original 7-by-3 room? Experiment with different room sizes. Predict how the size of the room affects where the laser beam ends up.

Hey, why don't you and the laser beam get a room?! Oh, right.

7. Evie's room has a floor tiled with 1-by-1 squares, as shown above. How many of these squares did Evie's laser beam cross over? Experiment with different room sizes. Predict how the size of the room affects the number of squares that are crossed.

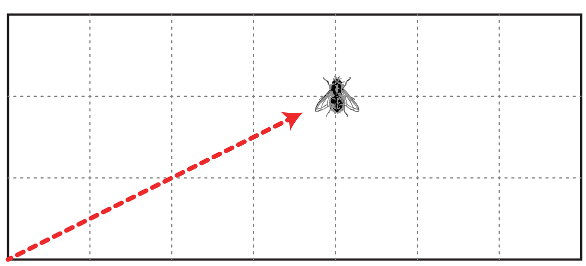
8. **a.** Find a fraction that you would consider to be close to $\sqrt{5}$.
b. Find a fraction that is even closer to $\sqrt{5}$.
c. Find a fraction that is yet even closer to $\sqrt{5}$.

Maybe they're just friends?

9. How many times does Evie's laser bounce off walls in the 7-by-3 room before hitting a corner? Find a general rule for the total number of bounces in an m-by-n room.

10. Jane notices there is a bug hanging out in Mary's 7-by-3 room, so she tries to hit it with Evie's laser. She misses but the laser beam continues along that path. Figure out stuff about the path of the laser.

The laser is pointed at $(4, 2)$. Assume, unlikely as it may seem, that there are no other bugs in Mary's room.



11. **a.** Find the smallest positive integer n so that $\frac{1}{n}$ is less than the value of $|\frac{22}{7} - \pi|$.

b. What could this information be useful for?

12. Suppose a and b are positive integers with no common factor greater than 1. In how many places will a straight path from $(0, 0)$ with slope $\frac{b}{a}$ cross a lattice line before hitting the point (a, b)?

$x = 3$ and $y = 4$ are lattice lines. Golden Corral has a lettuce line.

13. If a and b are positive integers with GCD d and LCM m, prove that

$$a \cdot b = d \cdot m$$

14. When Sweden plays Denmark, the scoreboard displays SWE-DEN. And, the letters that are unused spell DEN-MARK. Find other team names for which this is true.

Locals loved the last Super Bowl between New England and Philadelphia.

Tough Stuff

15. Betty's room is 3-dimensional and measures 3-by-7-by-11, with mirrors. She fires a laser beam from the lower left corner $(0, 0, 0)$ toward a speck of dust at $(1, 1, 1)$.

a. What corner of the room does the laser beam hit? Generalize to other sizes.

b. How long is the path of the laser beam? Generalize!

When the freakin' laser beam hits an edge that isn't a corner, it rebounds in two dimensions but continues the same direction in the third.

16. Like problem 9, determine the total number of bounces in a three-dimensional, L-by-W-by-H room.

17. Suppose a and b are irrational numbers so that

$$\frac{1}{a} + \frac{1}{b} = 1$$

Make a list of the multiples of a and b as decimals. Notice anything? Try to prove what you find.

Try picking a value for a, perhaps $\sqrt{2}$, to test the concept.

Day 2: Zealous Parade Planning

Opener

1. **a.** Here are some fractions you might have encountered in the Opener from Day 1:

$$\frac{311}{99}, \frac{289}{92}, \frac{267}{85}, \frac{245}{78}, \frac{223}{71}, \frac{201}{64}, \frac{179}{57}, \frac{22}{7}$$

Put these fractions in order of their closeness to π.

b. What fraction with denominator 150 or less is closest to π? Make a plan at your table to divvy up the work.

Yep, those are fractions, and we're definitely encountering them now.

Important Stuff

2. Calculate these.

   a. $(5+\sqrt{2})+(5-\sqrt{2})$

   b. $(5+\sqrt{2})\cdot(5-\sqrt{2})$

3. Find two numbers with the given sum s and product p.

s and p? Isn't the answer always 500?

   a. $s=10, p=25$

   b. $s=10, p=24$

   c. $s=10, p=23$

   d. $s=10, p=22$

   e. $s=10, p=21$

   f. $s=10, p=20$

   g. $s=10, p=1$

   h. $s=10, p=-1$

   i. $s=10, p=-299$

   j. $s=100, p=2451$

4. **a.** What is the closest that $\frac{n}{23}$, with integer n, can be away from $\frac{1}{3}$? (Express your answer as a fraction.)

   b. Explain why no fraction $\frac{n}{23}$ can be less than $\frac{1}{100}$ away from $\frac{1}{3}$.

   c. Find the fraction with the smallest possible denominator that's less than $\frac{1}{100}$ away from $\frac{1}{3}$, but isn't exactly $\frac{1}{3}$.

This is a nice problem!

5. **a.** Graph the line $y=\pi x$ on the attached graph paper. Do this as accurately as you can. Does your line pass through $(7,22)$?

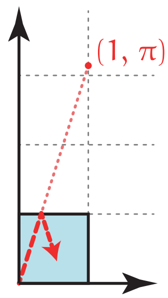
   b. Moving upward from $(0,0)$, what is the next lattice point (a point with integer coordinates) that the line $y=\pi x$ passes through?

(See the last page of today's problem set.)

Fun fact: according to Glencoe Algebra 1, this is not a line.

6. Ana lives in a 1-by-1 room, with mirror-covered walls, like Evie. She borrows Evie's laser and also shoots it from $(0, 0)$ in the direction toward $(1, \pi)$. Which corner of the room will the laser reach first?

There must have been some serious sale going on at Walls R Us. Oh, a going out of business sale, huh?



Neat Stuff

7. **a.** Use prime factorization to find the GCD of 672 and 1017.
   b. Do you see any limitations to this method?

Don't you wish there was a better way to find the GCD?

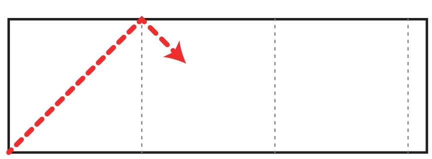
8. While doing Problem 1b today, you encountered some new fractions that are close to π. Include them in your list of fractions from Problem 1a. Look for any interesting patterns. Identify which fractions are less than π and more than π.

9. **a.** Find the smallest positive integer n so that $\frac{1}{n}$ can be less than $\left|\frac{22}{7} - \pi\right|$.
   b. Find the smallest positive integer n so that $\frac{1}{n}$ is less than the value of $\left|\frac{355}{113} - \pi\right|$.
   c. What could this information be useful for?

10. **a.** Find a fraction that you would consider to be close to $\sqrt{7}$.
   b. Find a fraction that is even closer to $\sqrt{7}$.
   c. Find a fraction that is yet even closer to $\sqrt{7}$.

The Chainsmokers would be proud.

11. Shana lives in a π-by-1 room with mirror-covered walls. She fires a laser from the lower left corner in the $45°$ direction. What happens?



12. Suppose $\sqrt{2} = \frac{a}{b}$ where a and b are positive integers.

- **a.** Write an equation relating a and b that doesn't have any fractions or square roots in it.
- **b.** What can you say about the number of twos in the prime factorization of a^2?
- **c.** What can you say about the number of twos in the prime factorization of $2b^2$?
- **d.** And then...?

No and then!

13. How many times does Evie's laser bounce off walls in her 7-by-3 room before hitting a corner? Separate bounces into top/bottom and left/right. Find a general rule for the total number of each type of bounce in an m-by-n room.

We were sad yesterday not to hear very many "boing" noises being made when the laser bounced off walls.

14. Suppose a and b are positive integers with no common factor greater than 1. In how many places will a straight path from $(0, 0)$ with slope $\frac{b}{a}$ cross a lattice line before hitting the point (a, b)?

x = -3 and y = 5 are lattice lines. The queue for a men's bathroom is a laddies' line.

15.
- **a.** Prove that for any three positive integers a, b, c,

$$\text{GCD}(a, b, c) = \text{GCD}\left(a, \text{GCD}(b, c)\right)$$

- **b.** Does the same tactic work for LCM?

16. For three numbers, the product of their GCD and LCM is equal to the product of the three numbers. Maybe.

If it's true, prove it; if it's false, prove it; if it's sometimes true, figure out when.

Tough Stuff

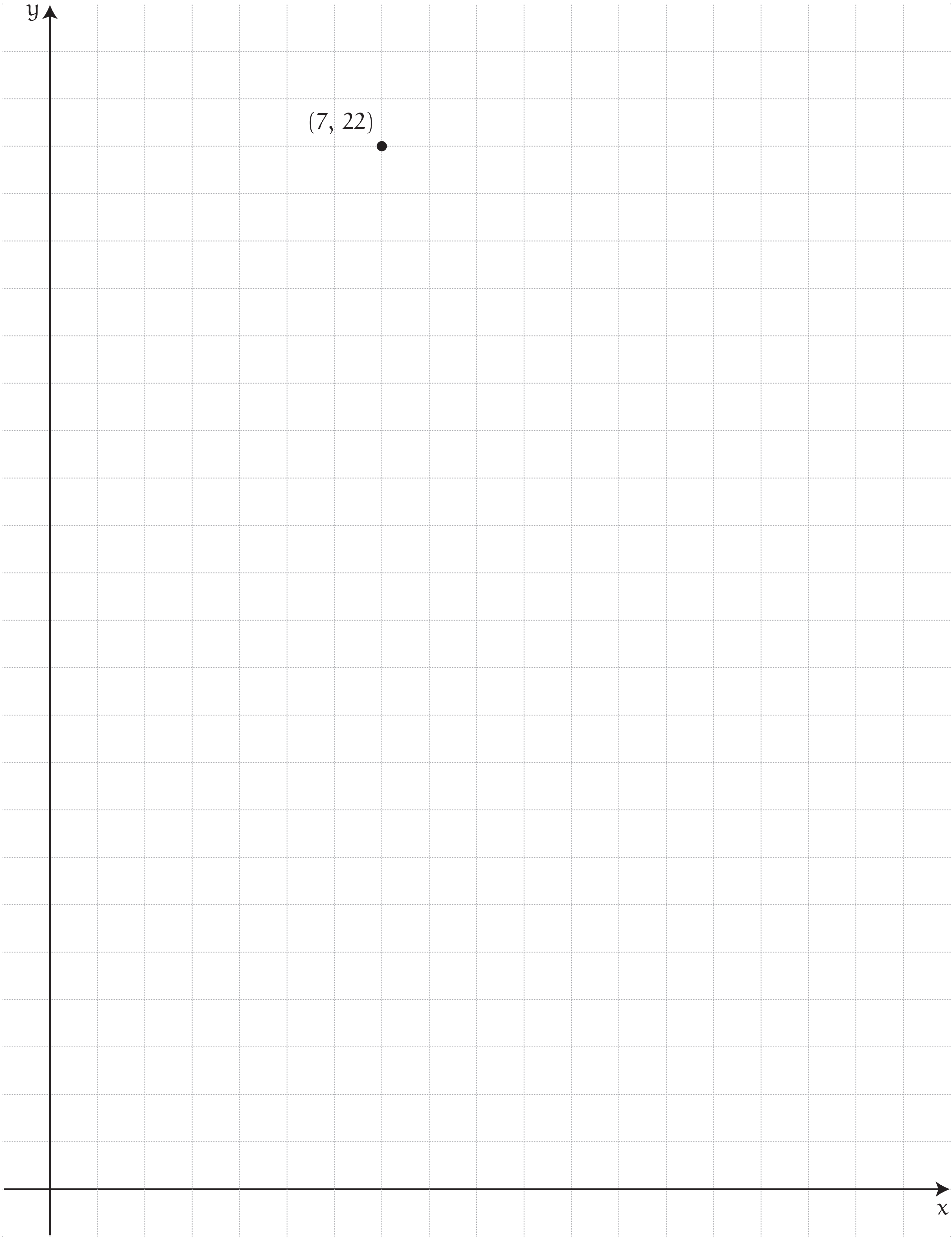
17. A laser beam is fired from $(0, 0)$. The beam's path ends if it comes within 0.001 units of any lattice point.

- **a.** Show that the beam must end, no matter what direction it is fired. *Or does it?*
- **b.** In what direction should the laser be fired to make the longest possible beam?

Hmm, I guess it does.

18.
- **a.** Calculate six digits of π starting at the 762nd decimal digit (the first decimal digit is 1).
- **b.** Calculate eight digits of π starting at the 79873884th decimal digit.

It's just like Problem 4, sort of.


y
(7, 22)
x

Day 3: Greatest Carnivorous Dinosaur

Opener

1. Westley takes a 200 mm-by-88 mm rectangle and starts cutting! He cuts away the largest possible *square* he can from one side of the rectangle. He doesn't stop until there is nothing left! Keep track of how many of each size square Westley made.

Fancy! We've provided you with your very own 200 mm-by-88 mm rectangle on a handout! You can cut or shade the squares—your choice.

2. What is the greatest common divisor of 200 and 88?

3. Here are some calculations that start with 200 and 88. What's going on here?

$$\begin{aligned} 200 &= 2 \cdot 88 + 24 \\ 88 &= 3 \cdot 24 + 16 \\ 24 &= 1 \cdot 16 + 8 \\ 16 &= 2 \cdot 8 \quad + 0 \end{aligned}$$

Important Stuff

4. Draw a 16-by-5 rectangle on some grid paper and Westley it up: remove the largest square that you can from one side of your rectangle. Keep removing the largest possible square until there is nothing left.
   - **a.** How many of each size square did you remove?
   - **b.** How big was the final square?
   - **c.** What is the GCD of 16 and 5?
   - **d.** Write calculations like the ones in Problem 3 for 16 and 5.

Careful! We are in the Old West. If you say "draw" too loudly, there might be trouble. This shouldn't be a problem before noon.

5. Repeat Problem 4 with a 28-by-12 rectangle.

6. Write each improper fraction as a mixed number in lowest terms.
   - **a.** $\dfrac{16}{5}$
   - **b.** $\dfrac{28}{12}$

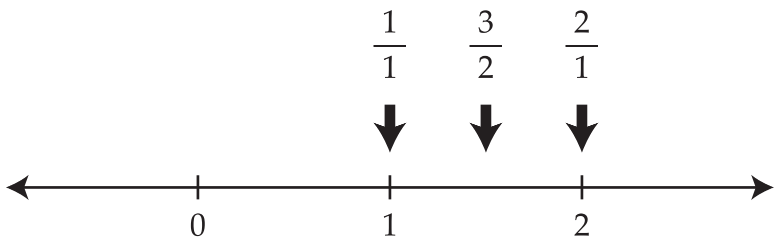
An improper fraction might set off illegal fireworks that land in other people's hot tubs.

7. Start with 0 and 1, then keep adding two terms to get the next:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

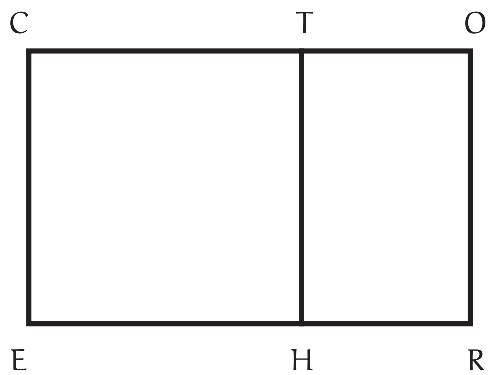
Use consecutive Fibonacci numbers (value/previous value) to make fractions and place these fractions on this number line. Keep placing! What do you notice?

Eventually these numbers start multiplying like rabbits. Wait, no, they're adding like rabbits.



8. You can take any non-square rectangle and chop a square out of it! In the diagram below, THEC is a square chopped out of rectangle CORE.

A *scaled copy* is like Ant-Man. Or there's a Wasp now? Shouldn't there only be half of them now? Anyway.



a. Suppose $EH = 2$ and $HR = 1$. Is rectangle ROTH a scaled copy of the original rectangle CORE?

b. Suppose $EH = 3$ and $HR = 2$. Is ROTH a scaled copy of CORE this time?

c. What if $EH = 5$ and $HR = 3$? Well, shoot.

d. If $EH = x$ and $HR = 1$, write a proportion that would have to be true if ROTH is a scaled copy of CORE. You don't have to solve it.

This one doesn't work either?? EH.

Tough Stuff

21. What's interesting about the fraction $\frac{100}{9899}$? Can you find any other super-interesting fraction friends?

No! No! It's too soon for the finale!

22. **a.** Westley uses his square-cutting method on a 6-by-5 rectangle, ending up with 6 squares. Find a way to partition this rectangle using fewer than 6 squares, and prove that it can't be done with even fewer.

b. Partition the 200-by-88 rectangle using the fewest number of squares possible.

c. Write an algorithm that will partition a m-by-n rectangle into the fewest number of squares.

Okay, it's fine, maybe people won't even notice the finale happened in the middle.

(This will make a lot more sense if you watched the fireworks last night.)

Neat Stuff

9. **a.** Hey, what was that smallest (denominator) fraction within $\frac{1}{100}$ of $\frac{1}{2}$ that wasn't exactly $\frac{1}{2}$?

b. . . . and that smallest fraction within $\frac{1}{100}$ of $\frac{1}{3}$?

c. Do these answers have anything in common?

d. Find the smallest fraction within $\frac{1}{1000}$ of $\frac{1}{13}$.

Okay, phew, back to normal.

10. **a.** What is the GCD of 495 and 132?

b. What is the GCD of $(495 - 132)$ and 132?

c. What is the GCD of $(495 - 3 \cdot 132)$ and 132?

d. What happens if you work through the Opener's square cutting with a 495-by-132 rectangle?

11. Joshua starts with a rectangle that is exactly $\sqrt{3}$ by 1, then starts square cutting. What happens?

Assume that Joshua can cut really, really, really small things accurately.

12. Find the rational number $\frac{p}{q}$ with smallest denominator $q > 0$ such that $\frac{p}{q}$ is between $\frac{17}{92}$ and $\frac{5}{27}$.

13. Find an integer solution to each equation, or explain why a solution cannot exist.

a. $3x + 5y = 1$

b. $3x + 6y = 1$

c. $3x + 7y = 1$

d. $29x + 13y = 1$

e. $154x + 69y = 1$

f. $495x + 132y = 1$

14. Take the diagram from Problem 8 and cut another square from rectangle ROTH. If $EH = x$ and $HR = 1$, determine another proportion that would have be true for the smallest length of the new rectangle formed.

Behold the mighty Thor! . . . no, wait, it's just his brother ROTH. His brother is the God of Retirement Planning.

15. **a.** Prove that $\text{GCD}(a, b) = \text{GCD}(a - b, b)$.
b. Prove that if r is the remainder when you divide a by b, that $\text{GCD}(a, b) = \text{GCD}(d, b)$.
c. What does this mean for Westley's square cutting?

16. Peter has a room that is 22-by-7. And Peter has a room that is 7π-by-7.

Peter and Peter each fire lasers from one corner. Compare the trajectories of the lasers.

You can see more of their adventures on Nickelodeon, Saturdays at 8 pm, 7 Central.

17. Find the GCD of $x^4 - 3x^3 - 6x^2 + 29x - 21$ and $x^5 + 7x^4 + 10x^3 - 18x^2 - 27x + 27$.

18. Find the GCD of $89 + 18i$ and $70 + 9i$.

19. Take the work of Problem 9 and apply it to π and $\frac{22}{7}$: what is the smallest (denominator) fraction within $|\frac{22}{7} - \pi|$ of π? Then try again with π and $\frac{355}{113}$.

20. **a.** Find all the ways to write integers a and b, with $a \leqslant b$, so that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{4}$$

b. Find a rule in terms of n for the number of ways to write integers a and b, with $a \leqslant b$, so that

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{n}$$

Alright, fire up the Tough Stuff! Problems 21 and 22, go! Wait what?

Day 4: Least Carnivorous Megalodon

Opener

1. Write this number as an improper fraction in lowest terms.

$$2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2}}}$$

An improper fraction doesn't respect MC Hammer. FYI, the more common term for improper fraction is "fraction".

2. Do some Westley-style square cutting using a rectangle whose width is 1 and whose length is the number from Problem 1. What up with that!

Ooooo weeeeee, what up with that, what up with that!

Important Stuff

3. Find the GCD of 741 and 2553.

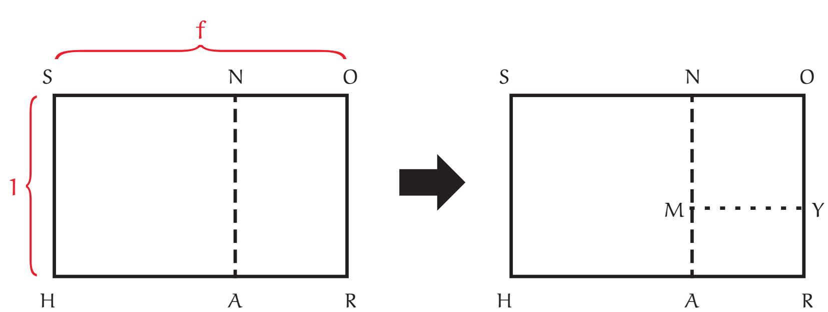
4. **a.** Explain how you know that, if a and b are any two positive integers, the Euclidean algorithm must end after a finite number of steps.

b. Westley wonders about cutting a rectangle with length $\frac{a}{b}$ and width 1, where a and b are positive integers. What can you say to Westley about the cutting?

Any number that can be written as $\frac{a}{b}$ where a and b are integers is called a *rational number*. Those that can't be written this way are just really unreasonable.

5. Rectangle SHARON has length $SO = f$ and width $SH = 1$, and it has the property that when square HANS is cut out of it, the resulting rectangle NORA is a scaled copy of SHARON. Another square is then cut out of NORA, leaving behind rectangle MARY.

"Did you see that? HANS got cut out!" "SO? SH."



a. Explain why MARY is a scaled copy of NORA.

b. Explain why $MARY$ is a scaled copy of $SHARON$.
c. Draw the next two cut lines.
d. How many more cut lines are there going to be? Explain how you know.

6. What do the results of Problems 4 and 5 say about the type of number f can be?

7. Calculate each of these.

a. $\left(\dfrac{1+\sqrt{5}}{2}\right) + \left(\dfrac{1-\sqrt{5}}{2}\right)$

b. $\left(\dfrac{1+\sqrt{5}}{2}\right) \cdot \left(\dfrac{1-\sqrt{5}}{2}\right)$

8. Find two numbers with sum $salt$ and product $pepa$.

Whatta number, whatta number, whatta mighty good number!

a. $salt = 7, pepa = 10$
b. $salt = 3, pepa = -10$
c. $salt = 13, pepa = 30$
d. $salt = 10, pepa = 21$
e. $salt = 100, pepa = -1469$
f. $salt = 1, pepa = -1$

If you get the last one right, you're golden. Problems about Spinderella may appear later.

Neat Stuff

9. **a.** Find the fraction with the smallest denominator that is between $\frac{2}{3}$ and $\frac{5}{7}$.
b. Find the fraction with the smallest denominator that is between $\frac{2}{3}$ and $\frac{7}{10}$.
c. Find the fraction with the smallest denominator that is between $\frac{2}{17}$ and $\frac{1}{8}$.
d. Describe a general pattern.
e. Find the fraction with the smallest denominator that is between $\frac{9}{37}$ and $\frac{8}{13}$.

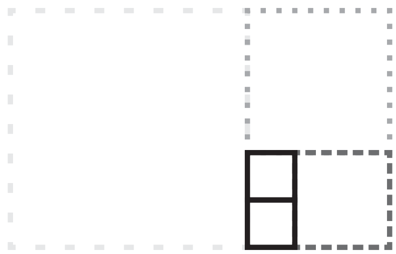
10. Melissa notices that when $\frac{22}{7}$ and $\frac{179}{57}$ are added together in the "wrong" way to get $\frac{201}{64}$, the resulting fraction happens to be in between the two original fractions. Investigate with other fractions to figure out the conditions when this happens. Prove your result.

Both $\frac{22}{7}$ and $\frac{179}{57}$ are *tart numbers*, because they are close to but not quite π.

11. Bryant is tired of removing squares and decides to add squares instead. He starts with a 1-by-1 square and adds another onto it, making a rectangle. He then adds a 2-by-2 square onto that, making a larger rectangle. Then a 3-by-3 square, making an even larger rectangle. Follow Bryant's steps on a piece of graph paper. Keep track of the sizes of squares you add and the ratio of the two sides of the entire rectangle (longer side to shorter side) at each stage. What do you notice?

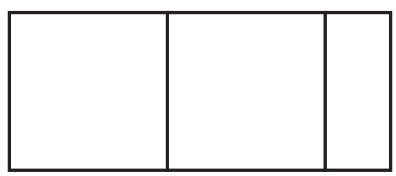
Bryant will know there are too many squares when Whoopi Goldberg or Paul Lynde or Ice Cube shows up. Yep, Ice Cube is a center square now.

Also follow Bryant's steps outside, as you walk toward the tent.



12. Roxanne cares little for the golden ratio, but she loves the *silver ratio*: it's the ratio of the length to width in a rectangle where cutting *two* squares from the long side leaves a scaled copy of the original rectangle.

This must be one of those Olympic sports where the lowest score wins.



a. What equation must the silver ratio satisfy?
b. What is the value of the silver ratio?

13. Determine the exact value of the *bronze ratio*, the ratio of the length to width in a rectangle where cutting *three* squares from the long side leaves a scaled copy of the original rectangle.

14. Create your own Westley-style square-removal problem that has the property that five different sizes of squares are removed.

Extra credit for creating a problem, swapping it with someone else's at your table, and solving their problem.

15. The number line is laid out from 0 to 1, with every fraction whose denominator is less than or equal to 10 marked with a point.

a. How many points are there?

b. The elevenths are added. How many now?
c. The twelfths are added. How many now?
d. What is the total sum of all the fractions, up to and including the twelfths?

Once you start talking about elevenths, it's usually either hobbit breakfasts, British Doctors, spinal taps, or girls from Indiana, but here we are anyway.

16. On the number line, $\frac{2}{5}$ and $\frac{3}{7}$ don't have any smaller-denominator fractions between them. But $\frac{3}{7}$ and $\frac{3}{5}$ do. Under what conditions will $\frac{a}{b}$ and $\frac{c}{d}$ have no smaller-denominator fractions between them?

17. Suppose a and b are irrational numbers so that

$$\frac{1}{a} + \frac{1}{b} = 1$$

Make a list of the multiples of a and b as decimals. Notice anything? Try to prove what you find.

Try $a = \sqrt{2}$ for one, but there are plenty of possibilities.

18. Find two irrational numbers that, when one is raised to the other, produce a rational number. Or, prove that it's impossible.

Tough Stuff

19. Take 9 squares, each of *distinct* integer side length, and pile them together to form a 33-by-32 rectangle.

This is an even worse version of Squares than the one Ice Cube is on. On the plus side, that show also features . . . Spinderella!

20. This is the list of all fractions with a maximum denominator of 4, in increasing order, from 0 to 1:

$$\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$$

Now take consecutive terms and multiply the denominators:

$$4, 12, 6, 6, 12, 4$$

What is the sum of the reciprocals of these numbers? Try it again with other maximum denominators. Prove that this will always work.

21. On Day 2, you found a very good approximation to π. Without using techmology, find the fraction with the lowest denominator that is a better approximation.

Yo, techmology is wack.

Day 5: Platypi Ease My Dreadful Aching Stomach

Opener

1. Michelle swears by her home remedy for colds, which consists of 2 tablespoons of ipecac syrup and 3 tablespoons of prune juice. The other Michelle swears by her home remedy for stomach aches, which consists of 5 tablespoons of ipecac syrup and 7 tablespoons of prune juice.

a. Poor Zach has both a cold and a stomach ache, so he makes both recipes and throws it together in a bowl. What is the ratio of ipecac syrup to prune juice in the resulting concoction?

Ah, the painful memories of Day 2 return.

b. Poor Stever has a really bad cold and a stomach ache, so he doubles the cold remedy recipe and throws it in a big bowl with the stomach ache remedy. What is the ratio of ipecac syrup to prune juice in the resulting concoction?

It's possible that this was all caused by Aunt Sally. Please excuse her.

c. Suppose x multiples of the cold remedy recipe are added to y multiples of the stomach ache remedy. Give some examples of what might happen. What is the smallest possible ratio of ipecac syrup to prune juice? The largest?

Important Stuff

2. a. Use mixture concepts to explain why

$$\frac{1}{3} < \frac{3}{8} < \frac{2}{5}$$

b. If $\frac{a}{b} < \frac{c}{d}$ with b and d positive, explain why

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

3. Armed with the knowledge of mixing, you can build any fraction! Start with pure prune juice on the left and pure ipecac syrup on the right:

You'll have to pretend the fraction $\frac{1}{0}$ is okay here. Thinking of it as pure prune juice . . . probably makes it worse.

$$\frac{0}{1} \qquad\qquad\qquad\qquad \frac{1}{0}$$

Mixing neighbors produces new fractions. Mixing produces $\frac{1}{1}$:

$$\frac{0}{1} \qquad\qquad \frac{1}{1} \qquad\qquad \frac{1}{0}$$

Mixing neighbors produces two new fractions:

$$\frac{0}{1} \qquad \frac{1}{2} \qquad \frac{1}{1} \qquad \frac{2}{1} \qquad \frac{1}{0}$$

What four new fractions can be added through mixing in the next phase?

No neighbor, no mix. No shoes, no service. No muss, no fuss. No more, no less. No harm, no foul. No pain, no gain. No credit, no problem. No justice, no peace. No woman, no cry. No diggity, no doubt. No tea, no shade. No guts, no glory. No runs, no hits, no errors. No I don't want, no scrubs. Mo money, mo problems.

4. Describe the mixing you will have to do to make $\frac{8}{5}$ and to make $\frac{13}{8}$. Don't bother describing the rest of the full set of numbers involved.

5. **a.** Rewrite this as a single fraction in lowest terms.

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}}$$

This type of expression is called a *continued fraction*. To be continued

b. Do some Westley-style square cutting using a rectangle whose width is the numerator of your answer above and whose length is the denominator.

c. How do the numbers from the Euclidean Algorithm connect with the numbers in the continued fraction?

. . . the expression involves multiple nested fraction bars. To learn more about continued fractions, check Wikipedia sometime after the next two weeks!

6. Consider a 121-by-38 rectangle. Keep chopping off the largest square possible until nothing is left.

a. How many squares did you chop off in each phase?

b. Write the continued fraction for $\frac{121}{38}$.

c. What is the GCD of 121 and 38?

7. Let's stop the fraction in Problem 5 after each step, and compare each value to $\frac{43}{30}$:

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2 + \frac{1}{3}}$$

What's going on?

I say hey.

Neat Stuff

8. On Day 2 you associated the number $\frac{22}{7}$ with the slope of the line passing through the origin and the lattice point $(7, 22)$.

a. Draw a coordinate plane illustrating the fractions $\frac{2}{3}$, $\frac{5}{7}$, 2, and $\frac{4}{3}$ as slopes, in a similar manner.

b. What is the fraction with smallest denominator that is between $\frac{2}{3}$ and $\frac{5}{7}$?

Stever, tell us all about it!

c. How can you use the visual of the coordinate plane to show the order of fractions?

9. **a.** Poor Lorena has both a cold and a stomach ache, so he mixes an *equal amount* of both remedies together. What is the ratio of ipecac syrup to prune juice in the resulting concoction?

Hey, just dropping in to let you know the answer to this question is *not* $\frac{29}{42}$. Yeah, ok bye.

b. Poor Scott has a really bad cold and a stomach ache, so he mixes two parts of the cold remedy with one part of the stomach ache remedy. What is the ratio of ipecac syrup to prune juice in the resulting concoction?

I'm beginning to think these concoctions might not be improving things for these poor folks.

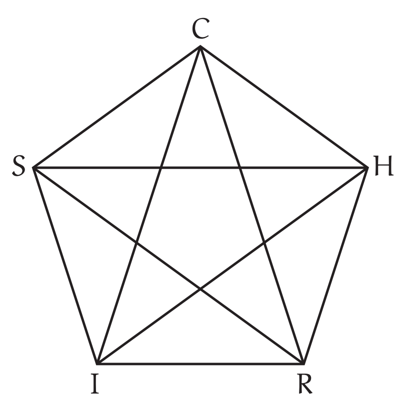
10. Here's a continued fraction:

$$3+\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{3}}}$$

If any of the threes changed to a four, the value of the continued fraction would change. Which would cause the most change? The least change?

Threes is so much better than 2048.

11. Pentagon CHRIS is regular.



a. Suppose $IR = 1$. What is the length of CI?

b. Look for other places where the ratio CI/IR appears in this figure.

Look for similar triangles instead of using trigonometry. You may want to label some more points, too.

12. Make a regular octagon in the coordinate plane using only lattice points, or explain why it can't be done.

A regular octagon has a high-fiber diet, and all its side lengths and angle measures are the same.

13. Make an equilateral triangle in the coordinate plane using only lattice points, or create a triangle using only lattice points that is a close to equilateral as possible.

14. Go back to the fractions you found on Day 1 as approximations for π. Determine how these fractions would be built using the mixing methods of Problem 3. Notice anything?

Baby go back! These fractions have pretty big bottoms, if you know what I'm saying. 24 over 36? Only if it reduces to two-thirds.

15. Start with any pair of integers from 0 to 9. Add them, taking only the units digit. Then keep going, adding the most recent two terms.

$$4, 8, 2, 0, 2, 2 \ldots$$

a. What happens?
b. How many different "chains" are there, and what is in 'em?

If 2 Chainz were here, he would guess at this problem, and he would be wrong.

16. What is the value of

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$$

17. When Collin plays Linda in the World Cup Final, what will the scoreboard display, and what will the unused letters in the names spell out?

Tough Stuff

18. Pack a 43-by-30 rectangle with as few squares as possible. How many squares is minimal?

19. Consider an infinite set of squares where square n has side length $\frac{1}{n}$. It turns out these squares have a finite total area! That means they can all fit in a 1-by-x rectangle. Find the smallest possible value of x.

If you really want to feel weird, compute the total perimeter of these squares.

20. Find the continued fraction for $\sqrt[3]{2}$.

Day 6: Favorite Otter Is Leopold

Opener

Here's a two-player game you can use to find numbers!

- The players decide on a target number. Let's say it's $\frac{11}{4}$.
- Player L starts at $\frac{0}{1}$ and Player R starts at $\frac{1}{0}$.
- The players "bad add" their numbers. One player moves to the new number, so that the target number is always between them.
- The players continue to "bad add" using their new locations.
- If the players "bad add" to reach the target number, the final move is taken by the player who last moved. Then high fives!

This game is best known as *Numbers With Friends*, and we're offering it to you free! Advertisements will be played after each number is found, hope that's okay.

Bad add a day, do dooooo da do doo.

And advertising.

1. **a.** Play the game with target number $\frac{11}{4}$. Who moves how many times, and in what order?

   b. Play the game with target number $\frac{4}{11}$. What happens now?

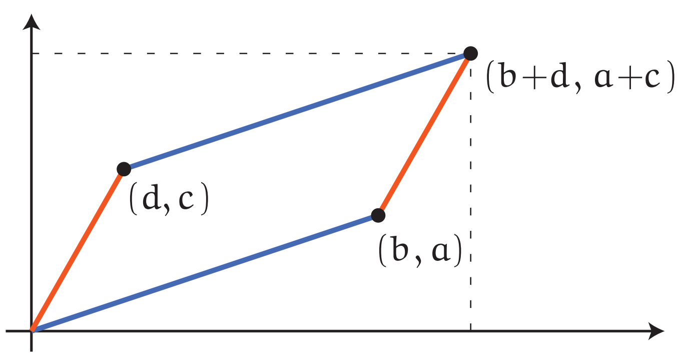
   c. Play the game with target number $\frac{43}{30}$.

This round of Numbers With Friends brought to you by the phone book! Yes, we still exist! Dial $\frac{4}{11}$ for more information.

Important Stuff

2. Find the area of this parallelogram. Do not continue to the next problem until you talk to someone else at your table who did this a different way.

And if you all did it the same way . . . find another way! High fives!



3. Use the Euclidean Algorithm to show that 11 and 4 don't share any common factors.

4. Write $\frac{11}{4}$ as a continued fraction.

5. Complete this to build the continued fraction for $\frac{150}{67}$.

Hi, five?

$$\begin{aligned}\frac{150}{67} &= 2 + \frac{16}{67} \\ &= 2 + \frac{1}{\frac{67}{16}} \\ &= 2 + \cfrac{1}{4 + \cfrac{3}{16}} \\ &= \cdots\end{aligned}$$

6. Do some Westley-style square cutting with a 150-by-67 rectangle and look for connections to Problem 5.

7. Play Numbers With Friends with the target number of $\frac{150}{67}$. Maybe you notice something, maybe you don't.

Maybe yes, maybe no. Maybe it's live, maybe it's Memorex. Maybe something bad, maybe something good. Maybe you should fly a jet, maybe you should be a vet. This round of Numbers With Friends brought to you by Maybelline.

8. Complete this to build the continued fraction for 2.76.

$$\begin{aligned}2.76 &= 2 + 0.76 \\ &= 2 + \frac{1}{1.315789\ldots} \\ &= 2 + \frac{1}{1 + .315789\ldots} \\ &= 2 + \cfrac{1}{1 + \cfrac{1}{3.166667\ldots}} \\ &= \cdots\end{aligned}$$

One way to calculate the reciprocal of a number using the TI-Nspire is $\boxed{\wedge}\,\boxed{(-)}\,\boxed{1}\,\boxed{\text{enter}}$.

9. **a.** Use the method of Problem 8 to find the continued fraction for $\sqrt{2}$.
b. Repeat for $\sqrt{3}$.

10. Play Numbers With Friends with the target number of $\sqrt{2}$. Describe what happens as best you can.

This round of Numbers With Friends brought to you by Energizer! The Energizer Bunny, it just . . .

Neat Stuff

11. **a.** What is the area of the parallelogram in Problem 2 using points $(b, a) = (3, 2)$ and $(d, c) = (7, 5)$?
b. What is the fraction with smallest denominator that is between $\frac{2}{3}$ and $\frac{5}{7}$?
c. Find another parallelogram with area 1 that involves $(0, 0)$ and either $(3, 2)$ or $(7, 5)$.

12. **a.** Draw a coordinate plane illustrating the fractions $\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, and $\frac{17}{12}$ as slopes.

b. Which of the fractions above is closest to $\sqrt{2}$?

13. Play Numbers With Friends with the target number of $\sqrt{3}$. Describe what happens as best you can.

This round of Numbers With Friends brought to you by the Ultramagnetic MCs, Johnny Mafia, Octave Minds, and Raekwon.

14. Consider the numbers

$$x = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4}}} \qquad \text{and} \qquad y = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{5}}}$$

What are the continued fractions for $2x$ and $2y$?

15. Find another fraction that is closer to $\sqrt{2}$ than any of the fractions listed in Problem 12. Then another.

16. Remember Roxanne's silver ratio in Problem 12 on Day 4?

You don't have to put on the silver light.

a. *Without doing any calculations*, determine the continued fraction for the silver ratio.

b. Find a fraction that approximates the silver ratio using some portion of its continued fraction.

17. **a.** Find the value of x if

$$x = 4 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}$$

b. What would a Numbers With Friends game look like if the target number was x?

This round of Numbers With Friends brought to you by Final Deployment 4: Queen Battle Walkthrough. You know the one, with Blair Trigger.

18. Find the value of x if

$$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

19. Generalize a pattern based on the results of Problems 17 and 18.

20. **a.** Natalie claims that when Westleying a m-by-n rectangle, where m and n are whole numbers, the last step should never be to remove a single square. Is that possible? Why or why not?

b. The other Natalie claims that a finite continued fraction can always be written in such a way that the final fraction isn't $1/1$. Is she right?

c. Look for a connection to Numbers With Friends.

21. **a.** Write this as a single fraction in lowest terms:

$$4+\cfrac{1}{0+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}}$$

b. Now find the continued fraction for the answer in part a. What? Maybe Numbers With Friends can help explain this.

22. We're been saying "the continued fraction", assuming that it's unique. Under what conditions is the continued fraction representation for a rational number unique?

23. **a.** Write the continued fraction for $\frac{80143857}{25510582}$. Use a calculator only to add, multiply, and subtract integers.

b. Now calculate the continued fraction for the same number, but first convert the fraction to a decimal. What happens? Why do you suppose this happens?

Tough Stuff

24. Find the value of

$$1+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{7+\dots}}}$$

25. Let n be prime.

High five^2?

a. For what n will the continued fraction representation for $\sqrt{n}$ have a repeating period of 1?

b. . . . a period of 2?

c. . . . a period of 3?

Day 7: Slurpees Slurpees Slurpees

Opener

1. Everyone in your group will work together on this problem! On graph paper, each person will draw a polygon where all its vertices are lattice points. For each polygon that is created, calculate

A = the area of the polygon

I = the number of lattice points completely inside the shape

B = the number of lattice points on the boundary of the shape

Collect enough examples in your group until you can come up with a relationship between A, I, and B.

A *lattice point* is on the grid and has integer coordinates. You decide how many sides the polygon has and how big it is. Big polygons are more interesting but more difficult to deal with, so **please start with simpler polygons**.

Important Stuff

2. A parallelogram is drawn on the coordinate plane. Its area is 1, and all its vertices are lattice points. One vertex is the origin $(0, 0)$ and the others are all in Quadrant I.

a. How many interior points and boundary points does this parallelogram have?

b. Give some examples of the possible other coordinates of this parallelogram.

Let's all agree to just start calling this the Alpha Quadrant. We can change the world.

3. juLie and maRy are playing Numbers With Friends. juLie controls the fraction on the left and moves 5 times, then maRy, who is controlling the fraction on the right, moves 2 times.

a. What numbers are they at at this point in the game?

b. Write this as a single fraction:

$$5 + \cfrac{1}{2 + \cfrac{1}{n}}$$

c. Julie finishes the game, taking n steps and landing at the target number. Determine the target number if $n = 1, 2, 3$.

This round of Numbers With Friends brought to you by Slurpee. Did you know you can bring your own cup and save? For more information, visit `bit.ly/pcmi_byoc`.

4. **a.** Perform the Euclidean Algorithm (or square cutting method) on the numbers 7 and 11. Yes, 7 is first.

b. Write out the continued fraction for $\frac{7}{11}$.
c. Use the continued fraction to predict the moves required to reach $\frac{7}{11}$, then verify it.

Walk out the door, turn left on Prospector, walk 3 blocks, look to your left.

5. For any two fractions $\frac{\clubsuit}{\diamondsuit}$ and $\frac{\heartsuit}{\spadesuit}$, the Funky Cross Thing that Robert discussed yesterday is defined as $\diamondsuit\heartsuit - \clubsuit\spadesuit$. Prove that when you replace either of two fractions with its "bad add," the Funky Cross Thing involving the new pair of fractions is the same as that involving the old pair of fractions.

The Funky Cross Thing is also a remix of Tone Loc's two hit songs.

6. Use a piece of grid paper to carefully draw a parallelogram using the origin, $(1,5)$, $(2,11)$, and the point in the Alpha Quadrant that completes that parallelogram.

Show me the parallelogram!! Quoth the parallelogram: "Fourth point, you complete me."

a. Use your formula from Problem 2 on Day 6 to calculate the area of this parallelogram.
b. According to your result from Problem 2, how many lattice points are enclosed by this parallelogram? Check to make sure.
c. Using this diagram, explain why $\frac{5}{1}$ "bad added" with $\frac{11}{2}$ must produce the fraction with smallest denominator that is in between both numbers.

Tonight on SportsCenter: Bad-add-ah, bad-add-ah.

7. eLeanor and spenceR play Numbers With Friends with the target of the golden ratio, $\phi = \frac{1+\sqrt{5}}{2}$. Who takes turns, and in what order? What else do you notice?

This round of Numbers With Friends brought to you by Golden Grahams cereal. 21 out of 34 people agree, Golden Grahams is phinomenal!

8. Simplify each of these as much as possible.

a. $(3+2\sqrt{2})(3-2\sqrt{2})$
b. $(10-7\sqrt{2})(10+7\sqrt{2})$
c. $(a+b\sqrt{2})(a-b\sqrt{2})$

9. Simplify each of these to the form $a \pm b\sqrt{2}$ where a and b are integers.

Rationalize those denominators!

a. $\dfrac{7+4\sqrt{2}}{3+2\sqrt{2}}$

b. $\dfrac{8-5\sqrt{2}}{10-7\sqrt{2}}$

c. $\dfrac{8-5\sqrt{2}}{5+3\sqrt{2}}$

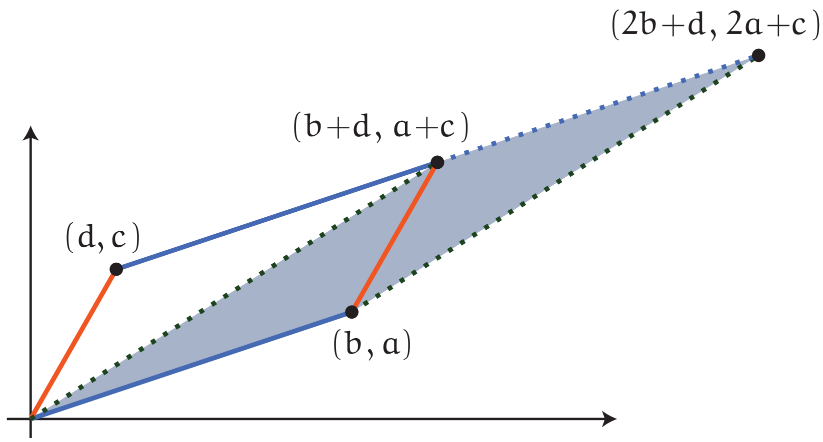
Neat Stuff

10. What are your ideas for future problem set titles?

11. Play Numbers With Friends . . . with a friend, of course. No specific target number, just make a few moves. After each move, verify the results of Problem 6 by constructing parallelograms from your fractions.

But you say, they're just a friend. This round of Numbers With Friends brought to you by Icy Drink! Next time you're in Oklahoma City, go to 7-Eleven and try to buy a Slurpee. You'll find there aren't any, but a suspiciously identical product called Icy Drink is! Stop by today.

12. Here's the same parallelogram from Day 6, but with an additional parallelogram!



a. Explain why the points (d, c), $(b + d, a + c)$ and $(2b + d, 2a + c)$ are collinear.

b. Explain why the area of the shaded parallelogram is equal to the area of the original parallelogram formed by the origin, (b, a), $(b+d, a+c)$, and (d, c).

c. Connect this back to some of your work in today's Important Stuff.

13. Find the value of x if

$$x = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}}$$

This problem is Manhattan friendly. What up, two-one-two!

14. **a.** Write the continued fraction for $\frac{80143857}{25510582}$. Use a calculator only to add, multiply, and subtract integers.

b. Now calculate the continued fraction for the same number, but first convert the fraction to a decimal. What happens? Why do you suppose this happens?

And we'll all turn float on, okay?

15. **a.** Generalize Pick's Theorem (the result in Problem 1), to polygonal shapes with a polygonal hole. All vertices, including the ones at the boundary of the hole, are at lattice points.

b. Generalize further to polygonal shapes with n polygonal holes.

The same rules apply: you get the total number of interior points and boundary points for the entire figure. In this case, no interior points would be inside the hole.

16. According to Pick's Theorem, which regular polygons can have all their vertices at lattice points?

17. Suppose $n > 0$. Find the value of x, in terms of n.

$$x = \sqrt{n + \sqrt{n + \sqrt{n + \sqrt{n + \dots}}}}$$

Ahem, don't just declare x to be that big 'ol nested square root, either.

18. Find the value of

$$y = \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \sqrt[3]{6 + \dots}}}}$$

19. Happy Sum Day! It's 7/11/18 and $7 + 11 = 18$. Sum Days are great.

a. How many Sum Days are there in 2018?

b. How many Sum Days are there in this century, 2001–2100?

"Don't forget some nights!" Ok, fun.

Tough Stuff

20. Find the fraction with smallest denominator that is within $\frac{1}{10^6}$ of $\sqrt{5}$.

Please don't tear apart the room looking for it. You're not going to find it that way! Jeez all these problems are asking you to find stuff. Tell you what, if you do figure out the answer, write it on a slip of paper and hide it somewhere, then some other person can truly find the answer.

21. Find a solution with $b > 0$ to the equation

$$a^2 - 13b^2 = 1$$

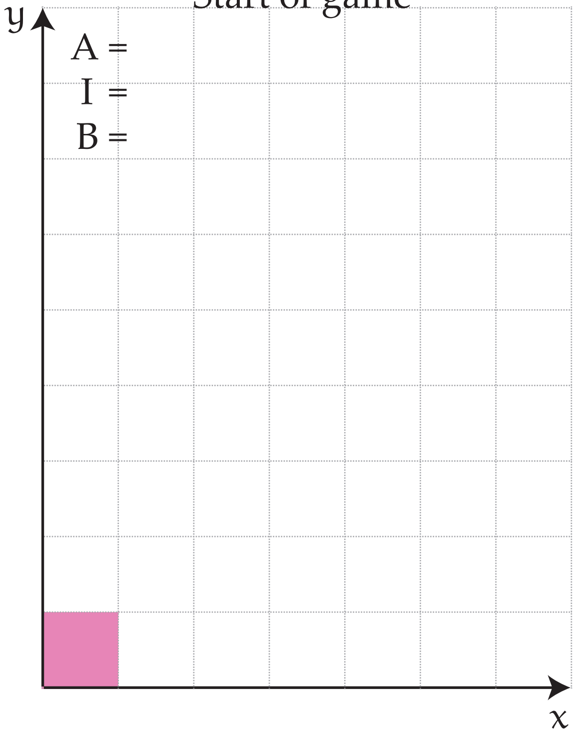
or show that no such solution can exist.

22. Find the value of

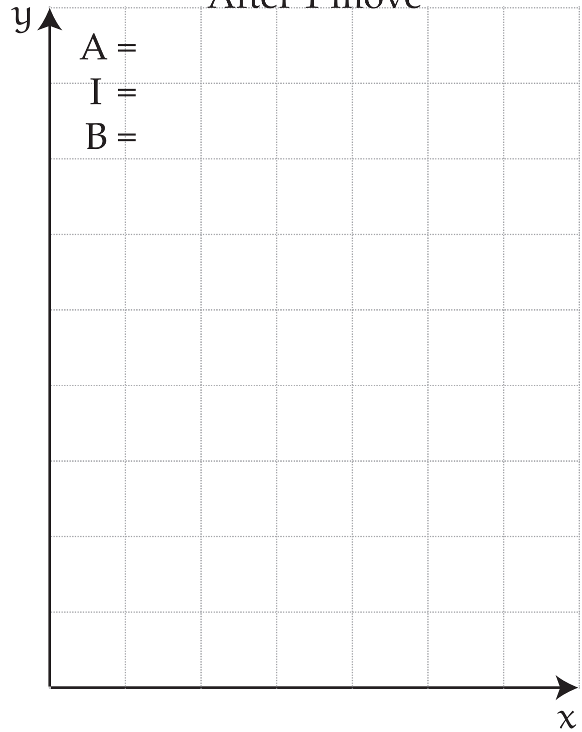
$$2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \dots}}}$$

23. Prove Pick's Theorem.

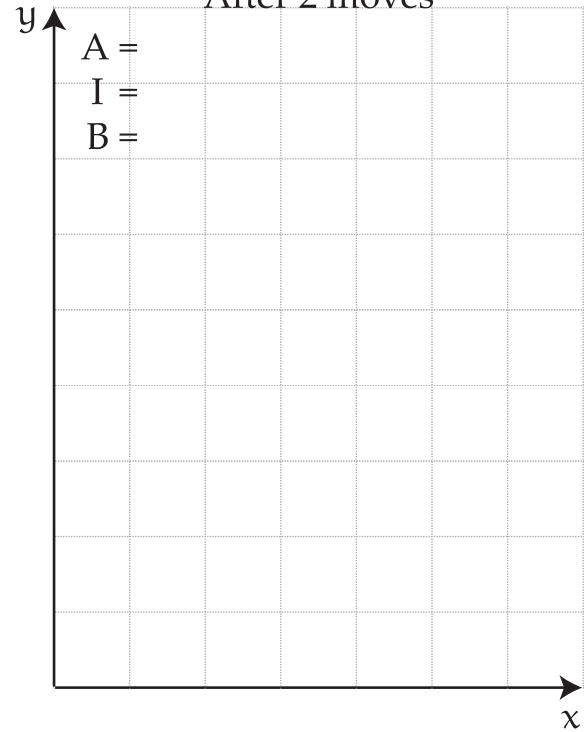
Start of game



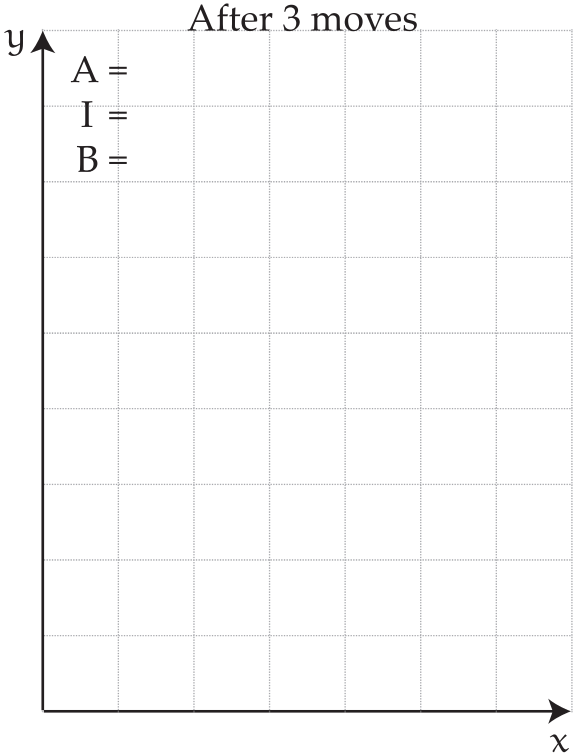
After 1 move



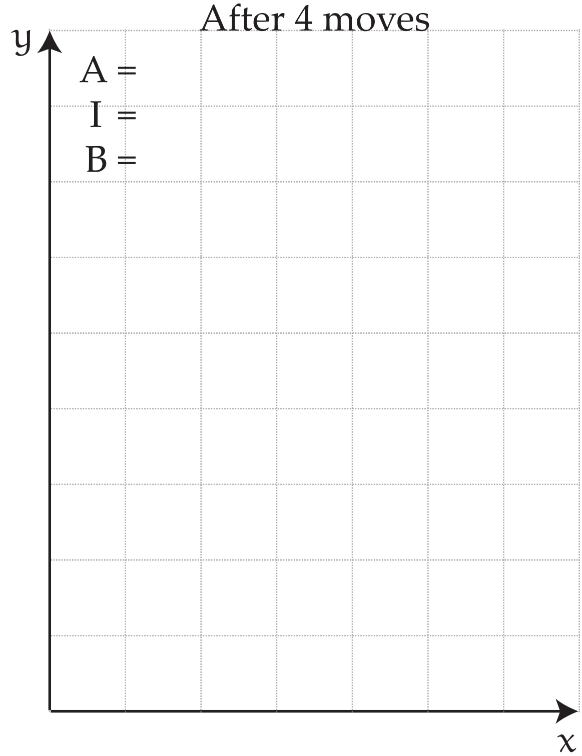
After 2 moves



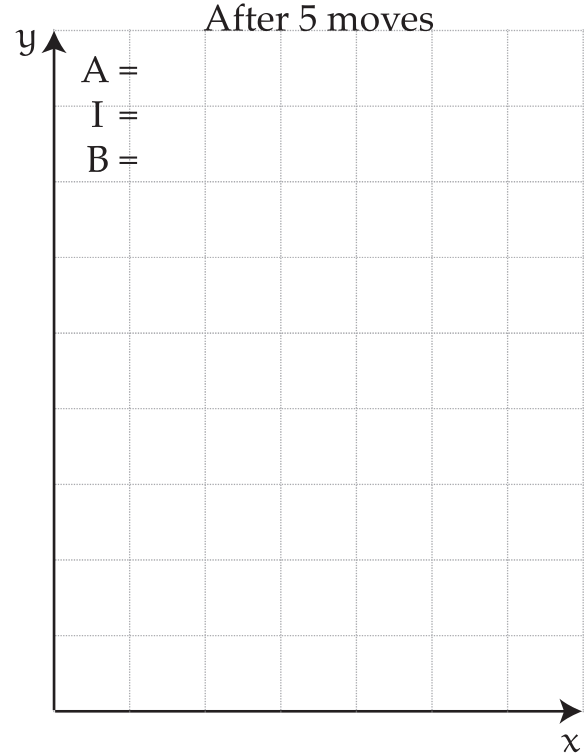
After 3 moves



After 4 moves



After 5 moves



Day 8: Corresponding Pants Create Turtle Chaos

Opener

micheLLe and peteR play a round of Numbers With Friends. They have purchased the new Box Upgrade™! The target number is $\frac{9}{7}$.

Be sure to emphasize those parts of their names whenever you see them from now on.

1. Play the game. Keep track of where L (left fraction) and R (right fraction) end up after each move with a table.
2. Write $\frac{9}{7}$ as a continued fraction.
3. Complete this Box keeping track of the locations of L and R after each player hands off to the other.

What's in the box??! Brad Pitt doesn't like boxes for some reason.

		# of L moves	# of R moves	# of L moves
			3	
$\frac{0}{1}$	$\frac{1}{0}$			$\frac{9}{7}$
Starting L	Starting R	Resulting L	Resulting R	Resulting L

Important Stuff

What's in the box??! The Cholula Buttery Jack, now for a limited time. This round of Numbers With Friends brought to you by Jack In The Box.

4. Play Numbers With Friends, targeting $\frac{43}{19}$. After each streak of moving the left (L) fraction or the right (R) fraction, keep track of your progress in the Box.

		# of L moves	# of R moves	# of L moves	# of R moves
$\frac{0}{1}$	$\frac{1}{0}$				
Starting L	Starting R	Resulting L	Resulting R	Resulting L	Resulting R

5. Rewrite each of these numbers as a single fraction.

$$2, \qquad 2+\frac{1}{3}, \qquad 2+\cfrac{1}{3+\cfrac{1}{1}}, \qquad 2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4}}}$$

What is the deal with that?

What is the deal with fractions? I mean are they one number, or two numbers? Make up your mind! And now they're continuing? They just don't stop! Somebody make them stop.

6. In Numbers With Friends, the starting left/right fractions are $\frac{0}{1}$ and $\frac{1}{0}$. These fractions correspond to the points $(1, 0)$ and $(0, 1)$, respectively. A parallelogram using these points along with the origin appears on the attached handout. Review the Numbers With Friends game you played in Problem 1. After each of the first five moves, draw the new parallelogram corresponding to the current left and right fractions. Calculate each parallelogram's area A, its number of interior lattice points I and its number of boundary lattice points B.

Hey, that's a square, not a parallelogram, you may be saying. If you feel that way, it's time to be more inclusive!

7. Play Numbers With Friends, targeting $\frac{73}{52}$. After each streak of moving the left (L) fraction or the right (R) fraction, keep track of your progress in the Box.

What's in the box??! Alice says there's a man in the box. In chains? Oh, guess not.

		# of L moves	# of R moves	# of L moves	# of R moves
					10
$\frac{0}{1}$	$\frac{1}{0}$				$\frac{73}{52}$
Starting L	Starting R	Resulting L	Resulting R	Resulting L	Resulting R

8. **a.** In the Box for Problem 7, describe how $\frac{73}{52}$ is built as a "bad add" from the previous two fractions.

b. Look back at Boxes and try to establish a pattern.

Sure we know why 6 is afraid of 7, but why did 7 eat 9? . . .

Neat Stuff

9. Simplify each of these as much as possible.

a. $(7 + 5\sqrt{2})(7 - 5\sqrt{2})$

b. $(5 - 3\sqrt{2})(5 + 3\sqrt{2})$

c. $(a + b\sqrt{2})(a - b\sqrt{2})$

10. Simplify each of these to the form $a \pm b\sqrt{2}$ where a and b are integers.

Please work this problem without techmology! What can you multiply the top and bottom of each fraction by?

a. $\dfrac{4 + \sqrt{2}}{5 + 3\sqrt{2}}$

b. $\dfrac{40 - 29\sqrt{2}}{7 - 5\sqrt{2}}$

c. $\dfrac{40 - 29\sqrt{2}}{10 + 3\sqrt{2}}$

11. Find an integer solution to each equation, or explain why an integer solution cannot exist.

a. $9x + 6y = 1$
b. $9x + 7y = 1$
c. $43x + 19y = 1$
d. $72x + 52y = 1$
e. $73x + 52y = 1$
f. $161x + 72y = 1$

Because 7 was told to eat 3 squared meals every day.

12. **a.** What is the continued fraction for $\sqrt{5}$?
b. Play the first few moves of Numbers With Friends with the target number $\sqrt{5}$. Describe who moves, and how many times.
c. Build a Box for this game, and use it to . . . yeah, what is this useful for anyway?

Do not play Numbers With Friends with Walter White. He will insist that he is the one who moves.

13. Two continued fractions are identical except for one of their coefficients. How can you tell, without calculating their values, which continued fraction is larger?

14. Two continued fractions aren't identical at all, they might even have different lengths. How can you tell, without calculating, which continued fraction is larger?

15. **a.** After five moves of a Numbers With Friends game, what is the maximum possible denominator of either player's location?
b. Generalize to n moves.

Here we mean five total moves. If L takes all five, they end up at $\frac{5}{1}$ and $\frac{1}{0}$. Not the maximum!

16. When playing Numbers With Friends, after a streak of n moves of the left fraction, the ______ fraction effectively gets replaced by ______ "bad added" to ______. After a streak of n moves of the right fraction, the ______ fraction effectively gets replaced by ______ "bad added" to ______.

17. **a.** Build a Box for a Numbers With Friends game using target number $\frac{5763}{706}$.
b. Find an integer solution to $5763x + 706y = 1$.

What's in the box??! It was a half-hour kids' show on Australian TV, of course.

18. Revisit Day 2, when you found fractions that approximate π and put them in order of their distance from π. Instead, Mia defines the "score" of a fraction to be its distance from π multiplied by its denominator. If a fraction has a denominator that is 10 times bigger, it also would need to be 10 times closer to π to have the same score. Order your fractions by Mia's score.

19. Prove that if an infinite continued fraction has a repeating pattern, then its value is of the form $\frac{a+\sqrt{b}}{c}$ where a, b, and c are integers.

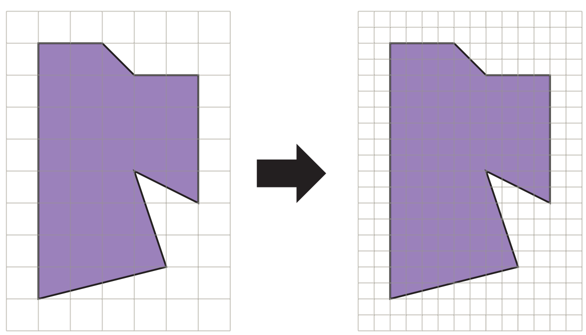
20. Find the value of

$$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{2+\dots}}}}$$

"I like you, too," says Darryl's watch. Or it likes their old stuff, anyway.

21. Remember Pick's Theorem, that result you found in Problem 1 on Day 7? Ashlyn wonders what happens to the number of boundary and interior points when the polygon is doubled in size. Create some examples and figure out what can happen.

If you prefer, you can think of the grid as getting half-sized.



22. What's your go-to karaoke song?

Tonight after the Estimathon, in the theater!

Tough Stuff

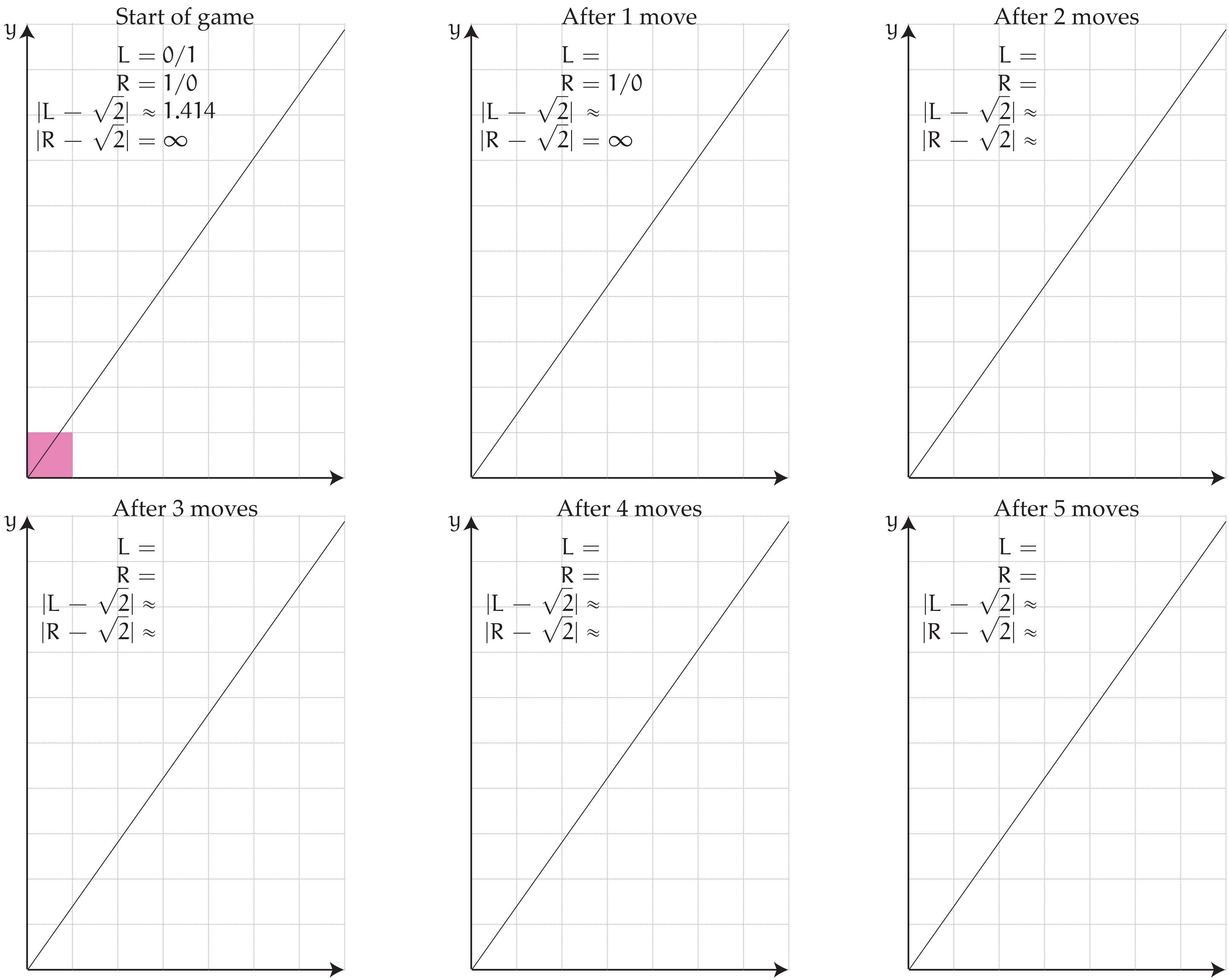
23. Which square numbers are also triangular numbers?

24. Find the value of

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\dots}}}$$

25. Compute the continued fraction for the "Euler number" e. Use the continued fraction to find some excellent fractional approximations to e, and explain why that continued fraction works.

What's in the box??! It's Brad Marchand, 2 minutes for slashing.



Start of game
y
$L = 0/1$
$R = 1/0$
$|L - \sqrt{2}| \approx 1.414$
$|R - \sqrt{2}| = \infty$
After 1 move
y
$L =$
$R = 1/0$
$|L - \sqrt{2}| \approx$
$|R - \sqrt{2}| = \infty$
After 2 moves
y
$L =$
$R =$
$|L - \sqrt{2}| \approx$
$|R - \sqrt{2}| \approx$
After 3 moves
y
$L =$
$R =$
$|L - \sqrt{2}| \approx$
$|R - \sqrt{2}| \approx$
After 4 moves
y
$L =$
$R =$
$|L - \sqrt{2}| \approx$
$|R - \sqrt{2}| \approx$
After 5 moves
y
$L =$
$R =$
$|L - \sqrt{2}| \approx$
$|R - \sqrt{2}| \approx$

Day 9: Several Old Hens Collaborate And Hastily Take Over Amazon

Important Stuff

1. **a.** Use the Euclidean Algorithm on 3317 and 1508.

   b. Use your work to help you fill out this Box from a game of Numbers With Friends targeting $\frac{3317}{1508}$.

This round of Numbers With Friends sponsored by Friday the 13th Part XLVII, Jason Goes To Math Camp.

		# of L moves	# of R moves	# of L moves	# of R moves
$\frac{0}{1}$	$\frac{1}{0}$				$\frac{3317}{1508}$
Starting L	Starting R	Resulting L	Resulting R	Resulting L	Resulting R

   c. Without doing any calculations, write down the continued fraction for this number.

Opener

2. **a.** In the style of Problem 8 from Day 6, calculate the continued fraction for π. Feel free to stop anytime after six levels.

   b. Daniel wants to know what fraction with denominator 10 or less is closest to π.

   c. Daniel wants to know what fraction with denominator 1,000 or less is closest to π.

   d. Daniel wants to know what fraction with denominator 10,000 or less is closest to π.

Only whole numbers are allowed for numerator and denominator. This is what is known as a "fraction bar".

---

3. Numbers With Friends now allows irrational targets! Steven and Stephen are playing, targeting $\sqrt{2}$.

   a. Use the attached handout to keep track of your first five moves. At the start, $L = \frac{0}{1}$ and $R = \frac{1}{0}$. These correspond to the points $(1, 0)$ and $(0, 1)$, respectively, because those points have slopes of L and R with the origin. Draw the parallelogram corresponding to your L and R after each step, and calculate how far L and R are from $\sqrt{2}$. A line with slope $\sqrt{2}$ has been provided to help you always keep $L < \sqrt{2} < R$.

Who will win? I suspect whatever happens, it will be Even Stev/phen.

b. After each streak of moving the L or R fraction, track your progress in this Box.

		# of L moves	# of R moves	# of L moves	# of R moves	# of L moves	# of R moves	
$\frac{0}{1}$	$\frac{1}{0}$							
Start L	Start R	L	R	L	R	L	R	

The creators of Numbers With Friends put lots of adds in the game to make sure players stay occupied for a loooong time. They're not even good adds.

4. Find a fraction with a three-digit denominator that is within $\frac{1}{10^6}$ of $\sqrt{2}$.

5. Explain why $\frac{355}{113}$ is a particularly good approximation for π.

6. Find integer solutions to each equation, or explain why no answers can exist.

Get by with a little help from your friends! All you need is your buddy. And math. But math is always your buddy.

a. $11x + 2y = 1$
b. $1102x + 201y = 1$
c. $3317x + 1508y = 1$
d. $22x + 7y = 1$
e. $333x + 106y = 1$
f. $355x + 113y = 1$

Neat Stuff

7. Find some fractions that are particularly good approximations for $\sqrt{269}$ and $\sqrt{373}$.

8. Dig in and find the continued fraction for various square roots, such as $\sqrt{10}$ or $\sqrt{15}$. Look for patterns and make conjectures about the continued fractions for other square roots.

Problem, you T-A-S-T-E-Y, tasty!

9. Consider fractions that are made from successive squares, such as $\frac{9}{4}$, $\frac{16}{9}$, $\frac{25}{16}$. Calculate their continued fractions. What pattern do you notice? Explain it?

10. **a.** Compute the continued fraction for e. What!
b. Compute the continued fraction for $\sqrt{e}$. What!!

This page has π *and* e? Hopefully there will be some sort of debate to settle this.

11. Take a shot at finding an integer solution to $6573x + 2915y = 1$. Then find them all.

Don't give away your shot! These solutions rise up. Without a solution there is onarchy.

12. Write this as a continued fraction without any zeros.

$$5 + \cfrac{1}{4 + \cfrac{1}{0 + \cfrac{1}{2}}}$$

Make sense of the answer in the context of a Numbers With Friends game.

Hey, I heard you like a wild zero, ooh-ooh-ooh.

13. Write these as simple fractions:

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}} \qquad 5 + \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

What's going on? Does it always happen? Experiment more! To thoroughly examine this problem, you'll need to figure out how to make sense of a continued fraction whose last coefficient is a 1 (see Problem 20 on Day 6) or a 0 (think of $\frac{1}{0}$ as ∞).

The problem writer has spoken: "I want it that way."

It's Corey's birthday, *and* it's Friday the 13th! This cannot be a coincidence.

14. We bet you didn't check *all* fractions with denominators less than 10,000 when you did Problem 2d. How do you know that your answer really is the closest to π among all other possible fractions?

You might find Problem 9 from Day 2 helpful.

15. What is the fraction with lowest denominator that is closer to π than the fraction you found in Problem 2d?

Oh hey, just dropping in again to let you know the answer to this problem does *not* have denominator larger than 30,000. Ok, bye.

16. Suppose the irrational number x has continued fraction

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

Define x_k to be the number created by using only a_0 through a_k and truncating after a_k. It's like stopping Numbers With Friends a little early.

a. Explain why the sequence $x_0, x_1, x_2, \ldots$ alternates between being below and above the target x.

b. Explain why x_k must be closer to x than any of the previous x_i.

Thinking of these problems in terms of Numbers With Friends can help!

17. Pick a prime p. Calculate the continued fractions for $\frac{1}{p}$, $\frac{2}{p}$, to $\frac{p-1}{p}$. Each of the continued fractions in this set will pair up nicely with another one in the set if you reverse the coefficients in its continued fraction. Investigate which fractions pair up nicely.

Make sure that you do Problem 20 on Day 6 first before you try this problem. This problem is really long but Mary has set it to run at 25% faster than normal.

18. **a.** Find Pythagorean triples where the hypotenuse and long leg are only 1 apart. Classify all such triples.

b. Find Pythagorean triples where the short leg and long leg are only 1 apart. Classify all such triples.

Can anybody find me . . . some triangles to love?

19. Find the value of

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\dots}}}}$$

20. If you pick two vectors from the origin, (b, a) and (d, c), the shape made from those vectors (and their sums) is a parallelogram. In three dimensions, the same concept determines a *parallelepiped* spanned by three vectors from the origin: (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

This could be the best math term ever. Parallelepiped! Say it three times: paralleleperhaps Michael Keaton will come out of the box, or a parallelepied parallelepiper might appear. A die made in this shape has parallelepips on it.

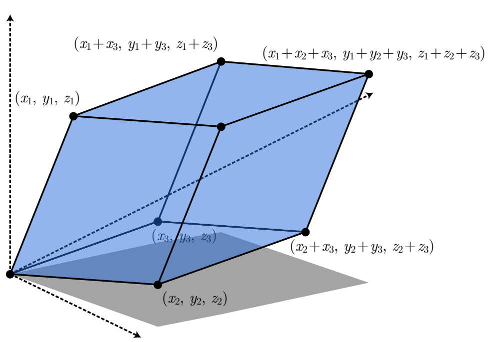


Figure adapted from `http://en.wikipedia.org/wiki/File:Determinant_parallelepiped.svg`.

Find the volume of the parallelepiped in terms of the nine variables. Try to do this geometrically, without relying on a formula.

This problem is shear madness, but still less madness than two people standing up and pretending to be Slim Shady at once.

Tough Stuff

21. Prove that if n is any integer, then $n^4 + 64$ is not prime.

22. In terms of k, find the number of factorable quadratics in the form $x^2 + ax + ka$.

For example, if $k = 2$, it works when $a = 8$ and $a = 9$, and maybe others.

Day 10: Without Lots of Gorillas

Opener

1. Using techmology, find as many lattice points as you can on the graph of each equation.

We love Desmos, but you can use any tool you like.

a. $y = \sqrt{6}x$
b. $y^2 - 6x^2 = 1$

Important Stuff

2. Without using techmology or the decimal value of $\sqrt{6}$, decide which is bigger: $\frac{5}{2}$ or $\sqrt{6}$? $\frac{7}{3}$ or $\sqrt{6}$?

Yo. Science. What is it all about? Techmology. What is that all about? Is it good, or is it wack?

3. Ahngelique has another take on Problem 2. She says that whether $\frac{5}{2}$ is bigger or smaller than $\sqrt{6}$ depends on whether $5^2 - 6 \cdot 2^2$ is positive or negative. Figure out how that works. What about $\frac{12}{5}$ and $\sqrt{6}$?

4. So, you just had a weekend. Do you remember how to play Numbers With Friends . . . targeting $\sqrt{6}$?

		# of L moves	# of R moves	# of L moves	# of R moves	# of L moves	# of R moves	
$\frac{0}{1}$	$\frac{1}{0}$							
Start L	Start R	L	R	L	R	L	R	

5. For each fraction $\frac{y}{x}$ in the Box above, calculate $y^2 - 6x^2$. What do you notice?

What's in the box??! These numbers. And the fractions in the second row are called *convergents*. You can call them convergents, you can call them box numbers, you can call them turning points, but you don't ever have to call them Mr. Johnson.

6. Why can't $y^2 - 6x^2$ ever be zero if x and y are nonzero integers?

7. Without using a calculator, determine the largest integer less than each of these.

a. $\sqrt{7}$ **b.** $\sqrt{19}$ **c.** $\sqrt{23}$

d. $\dfrac{\sqrt{7}+1}{2}$ **e.** $\dfrac{\sqrt{7}+1}{3}$ **f.** $\dfrac{\sqrt{19}+4}{3}$

g. $\dfrac{\sqrt{19}+2}{5}$ **h.** $\dfrac{\sqrt{19}+3}{2}$ **i.** $\dfrac{\sqrt{19}+3}{5}$

But wait, there's more!

j. $\dfrac{\sqrt{23}+4}{7}$ **k.** $\dfrac{\sqrt{23}+3}{2}$ **l.** $\dfrac{\sqrt{23}+3}{7}$

8. Multiply these using your brain!

a. $\left(\sqrt{7}+1\right)\left(\sqrt{7}-1\right)$
b. $\left(\sqrt{19}+2\right)\left(\sqrt{19}-2\right)$
c. $\left(\sqrt{23}-4\right)\left(\sqrt{23}+4\right)$

Braaaaaaains. Lately the zombies seem more interested in fighting plants, or just disappearing while a weird guy with a baseball bat ruins the show.

9. Rewrite these so denominators have no radicals:

a. $\dfrac{2}{\sqrt{7}+1} = \dfrac{2}{\sqrt{7}+1} \cdot \dfrac{\sqrt{7}-1}{\sqrt{7}-1} =$
b. $\dfrac{3}{\sqrt{19}-2} = \dfrac{3}{\sqrt{19}-2} \cdot \underline{\qquad\qquad} =$
c. $\dfrac{7}{\sqrt{23}+4} =$

10. Rewrite these so numerators have no radicals:

a. $\dfrac{\sqrt{7}-2}{1} = \dfrac{\sqrt{7}-2}{1} \cdot \underline{\qquad\qquad} =$
b. $\dfrac{\sqrt{19}-3}{5} =$
c. $\dfrac{\sqrt{23}-3}{7} =$

This is called *rationalizing the numerator*. Generally, this is used to annoy teachers. But it's also useful! Magic 8 Ball says "Ask Again Later".

Neat Stuff

11. **a.** Using any method you like, calculate the continued fractions for $\sqrt{7}$, $\sqrt{11}$, and $\sqrt{17}$.
b. Kai Sam says he noticed a relationship between the first coefficient and the last coefficient before the continued fraction repeats itself. What do you think?
c. Explore the continued fractions for $\sqrt{n}$ for other integer values of n.

In this case, techmology is good, not wack.

12. Calculate the value of each expression. Only use a calculator if you are desperate.

a. $(5+2\sqrt{6})(5-2\sqrt{6})$
b. $(22+9\sqrt{6})(22-9\sqrt{6})$
c. $(5+2\sqrt{6})(22+9\sqrt{6})$

Desperate like a 2005 housewife?

d. $(218 + 89\sqrt{6})(218 - 89\sqrt{6})$
e. $(22 + 9\sqrt{6})^2$
f. $(970 + 396\sqrt{6})(970 - 396\sqrt{6})$

13. Calculate the value of each expression. Only use a calculator if you are distraught.

Distraught, like a 2012 employee of *Desperate Housewives*? Wait, that show went for *180 episodes?*

a. $(4 + \sqrt{13})(4 - \sqrt{13})$
b. $(4 + \sqrt{13})^2$
c. $(29 + 8\sqrt{13})(29 - 8\sqrt{13})$
d. $29^2 - 13 \cdot 8^2$
e. $(4 + \sqrt{13})^3$
f. $220^2 - 13 \cdot 61^2$

14. It's three weeks into the baseball season. Tun notices that his favorite White Sox player's batting average is .279. How many at-bats and hits do they have?

Batting average is a decimal, hits divided by at-bats, displayed to the nearest thousandth. Assume the player has less than 60 at-bats.

15. Three consecutive integers have these continued fractions for their square roots:

$$5+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{10+\cdots}}}} \qquad 5+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{10+\cdots}}}} \qquad 5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{10+\cdots}}}}$$

(Note: All three of them have repeating continued fraction coefficients of the form $5, 1, *, 1, 10, 1, *, 1, 10, \ldots$.)

a. Without calculating any of them, put the numbers in increasing order.
b. Okay, now you can calculate them.

Geez, will you just tell me whether or not techmology is wack?

16. The fractions $c_2 = \frac{22}{9}$ and $c_3 = \frac{49}{20}$ are consecutive convergents to $\sqrt{6}$.

a. Calculate $|c_2 - c_3|$ as a fraction.
b. Why does this work?
c. Explain why $|c_2 - \sqrt{6}|$ must be less than $|c_2 - c_3|$.
d. Verify this also works for $c_3 = \frac{49}{20}$ and $c_4 = \frac{218}{89}$.

"Because math" is not an acceptable answer . . . but go ahead and write it down anyway, it's fun.

17. Use the work in Problem 16 to prove that for any convergent $c = \dfrac{p}{q}$ to $\sqrt{6}$, it's true that $|c - \sqrt{6}| < \dfrac{1}{q^2}$.

18. **a.** Find Pythagorean triples where the hypotenuse and long leg are only 1 apart. Classify all such triples.

b. Find Pythagorean triples where the short leg and long leg are only 1 apart. Classify all such triples.

Pythagorean triples are the most difficult part of hitting for the Pythagorean cycle.

19. The number 1 is both square and triangular.

a. Find the next three numbers that are both square and triangular, if they exist at all . . .

b. Describe a method that could generate as many such numbers as needed . . . if they exist at all.

20. Draw the line $x + y = 11$. Draw segments from $(1, 10)$ to $(0, 0)$, from $(2, 9)$ to $(0, 0)$, . . . , from $(10, 1)$ to $(0, 0)$. This creates a buncha buncha little triangles.

Yes yes, we know there are even more triangles, you, you, math teacher.

a. Josie counts the interior lattice points in each triangle, and suggests you do the same.

b. Explain why this worked.

c. Will this generalize to $x + y = n$ for any integer n?

21. Let's look at the first page with the target number of $\sqrt[3]{2}$.

a. Find as many lattice points as you can on the graph of $y^3 - 2x^3 = 1$.

b. Without using techmology, which is bigger: $\frac{5}{4}$ or $\sqrt[3]{2}$?

c. Play Numbers With Friends targeting $\sqrt[3]{2}$ and fill in the Box with at least eight fractions.

d. For each fraction $\frac{y}{x}$ in the Box, calculate $y^3 - 2x^3$. What do you notice?

I noticed that this problem is at the bottom of the last page, so it's probably really weird.

Tough Stuff

22. Hey, what about negative coefficients in continued fractions? What happens then?

23. Find the smallest integer N for which the continued fraction of $\sqrt{N}$ does *not* repeat, or prove that this is impossible.

24. Determine the equivalent to Pick's Theorem in three dimensions, finding the volume of a shape given its number of interior points and . . . some other stuff.

Day 11: Left Hand Shark

Opener

1. Find some lattice points on the graph of each equation. You may or may not need the actual graphs . . .

Actual graphs, actual size! What a terrible tagline for Graph-Man and The ASTC.

a. $y = \sqrt{7}x$
b. $y^2 - 7x^2 = 1$

Important Stuff

2. The fraction $\frac{5}{4}$ is not equal to $\sqrt{2}$. Compute each of these by hand.

The ratio $\frac{5}{4}$ is the number of fingers to the number of fingers.

a. $5^2 - 2 \cdot 4^2$
b. $(5 + 4\sqrt{2})(5 - 4\sqrt{2})$
c. $(5 + 4\sqrt{2})^2$
d. $57^2 - 2 \cdot 40^2$

3. Take a look at this file: `bit.ly/pcmisqrt7`

a. What happens at each step of the calculation?
b. The same sequence of steps keeps repeating. What are those steps, in order?
c. Based on the calculation, how do you know that the continued fraction for $\sqrt{7}$ will repeat? How many and which numbers are repeated?

4. Perform a similar calculation to Problem 3 to determine the continued fraction for $\sqrt{41}$.

Use arithmetic with square roots here, not calculator approximations.

5. On the next page is a Box for Numbers With Friends targeting $\sqrt{7}$. You already know the continued fraction for $\sqrt{7}$ so you don't have to play the game! Yay!

You're ready for the next version of the game: Numbers By Yourself. Aww.

a. Fill in the first row of the Box on the next page, then the second row.
b. Hey, so those fractions that you get by truncating the continued fraction at a particular place? They're called *convergents*. For each convergent $\frac{y}{x}$, calculate the value of $y^2 - 7x^2$ and write it in the third row.

Yes yes, you told us yesterday. They're also the fractions in the second row of the Box.

		# of L moves	# of R moves	# of L moves	# of R moves	# of L moves	# of R moves	
		2						
$\frac{0}{1}$	$\frac{1}{0}$					$\frac{37}{14}$		
Start L	Start R	L	R	L	R	L	R	
$y^2 - 7x^2$ value								

Neat Stuff

6. Find another solution $y^2 - 7x^2 = 1$ with $x > 40$.

When interviewed recently, Left Shark said he deliberately messed up, going for "maximum goofy". Left Shark, you are no Goofy.

7. Find a solution to $y^2 - 41x^2 = 1$ with $x > 0$.

8. Calculate.

a. $(8 + 3\sqrt{7})(8 - 3\sqrt{7})$
b. $(8 + 3\sqrt{7})^2$
c. $(32 + 5\sqrt{41})(32 - 5\sqrt{41})$
d. $(32 + 5\sqrt{41})^2$

9. Explore some connections involving Problem 8 and see what else you can discover.

Perhaps you'll discover that this is Left Shark Week.

10. What numbers are these?

a. $10 + \cfrac{24}{10 + \cfrac{24}{10 + \cfrac{24}{10 + \cdots}}}$ **b.** $10 - \cfrac{24}{10 - \cfrac{24}{10 - \cfrac{24}{10 - \cdots}}}$

Um . . . they're pretty clearly 10 and 24.

11. Determine the value of each continued fraction, each of which repeats its three coefficients.

a. $1 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}$ **b.** $3 + \cfrac{1}{2 + \cfrac{1}{1 + \cdots}}$ **c.** $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cdots}}$

12. Sean's spidey senses tingle when he notices that if he directly adds the coefficients of the continued fractions for $\sqrt{2}$ and $\sqrt{5}$, he gets the continued fraction for $\sqrt{10}$. Oh, and $2 \times 5 = 10$. What up with that!

Maybe something! Maybe nothing! Maybe a whole lot of nothing! I say ooooh eeeee, what up with that, what up with that!

13. Deb says one student always turns in decimals instead of fractions, but she can figure out what fraction they meant. For each decimal, find the fraction of smallest denominator that matches the fraction.

a. 0.157
b. 0.6316
c. 0.40238

That last one got the kid sent to the office.

14. **a.** Suppose x is represented by a continued fraction that eventually repeats. What sorts of numbers can x be?

b. Suppose y is represented by a continued fraction that perfectly repeats, like the ones in Problem 11. What sorts of numbers can y be?

The continued fractions for square roots generally do *not* perfectly repeat. $\sqrt{2}$ doesn't perfectly repeat, but it almost does . . .

15. Take some more time looking at the period of each $\sqrt{n}$ and patterns in the coefficients. There's a lot to see!

16. The *Brocot sequence* starts with $\frac{0}{1}$ and $\frac{1}{1}$ at Level 0. Each new level includes all "bad add" fractions that sit between the previous level.

a. Write out the Brocot sequence of Level 3, of Level 4.
b. Explain why $\frac{12}{43}$ will be part of the Brocot sequence. When and where will it appear?
c. Take all the pairs of consecutive terms of the Brocot sequence of level 4, and multiply their denominators. This gives another sequence of numbers. Add the reciprocals of all these numbers; what happens, and why?

This sequence is named after an incident where a fraternity ran out of beds.

17. The *Farey sequence* of order n is all the fractions from $\frac{0}{1}$ to $\frac{1}{1}$ with denominator less than or equal to n, written in increasing order.

a. When and where will $\frac{12}{43}$ appear in the Farey sequence?
b. Take all the pairs of consecutive terms of the Farey sequence of order 10, and multiply their denominators. This gives another sequence of numbers. Add the reciprocals of all these numbers; what happens, and why?

This sequence is named after an incident where a Canadian gave very long driving directions.

18. Here are lots of continued fractions for various $\sqrt{n}$: `bit.ly/pcmisqrt`. Rachel suggests we look at $\sqrt{19}$. She says that at each stage, there is a number S from which we take the largest integer. Rachel claims that the value of S in any stage is completely determined by the value of S at the previous stage. Is she right? Explain why or why not.

Why did Kellogg's stop making Product 19? Because there was only one way to make it.

19. Continuing with our study of $\sqrt{19}$ ... Notice how every instance of S looks like

$$\frac{\sqrt{19}+T}{B}$$

where T and B are non-negative integers. Let a be the largest integer that is subtracted from S. Fill out this Mega Box.

We ran out of ideas for what to call these numbers, so we decided to use T for the number in the top and B for the number in the bottom. Or maybe they're just named after Tom Brady.

Stage n	**0**	**1**	**2**	**3**	**4**	**5**	**6**
S_n	$\sqrt{19}$			$\frac{\sqrt{19}+3}{2}$			
T_n	0						
B_n	1						
a_n	4	2					

Look for patterns in the Mega Box, particularly relationships between numbers in columns n and $n+1$.

20. Let $S_n = \dfrac{\sqrt{19}+T_n}{B_n}$. We get S_{n+1} by subtracting a_n from S_n, then doing stuff to it.

What stuff, you ask? Some important stuff, some neat stuff, and some tough stuff.

a. First, figure out what T_{n+1} has to be in terms of T_n, B_n, and a_n.

b. Now figure out what B_{n+1} has to be in terms of ...

21. Once you have recursive formulas for T_n, B_n, and a_n (conjectures are great!), test it out for another $\sqrt{n}$ continued fraction that you haven't yet studied but is in `bit.ly/pcmisqrt`.

Tough Stuff

22. Using continued fractions, generate some Pythagorean triples that are very, very close to being 30-60-90 right triangles.

One such triangle is *not* 31-61-91.

Day 12: Radical Hippopotamus Strategy

Opener

1. **a.** Find a bunch of lattice points on the graph of $y^2 - 2x^2 = 1$.
   b. Calculate $(3 + 2\sqrt{2})^2$.
   c. Find more lattice points on the graph of $y^2 - 2x^2 = 1$.

Eat a bunch of Crunch & Munch? Or maybe Shakey's Bunch of Lunch? Just a hunch.

Important Stuff

2. Calculate each of the following.
   a. $(7 + 3\sqrt{5})(7 - 3\sqrt{5})$
   b. $7^2 - 5 \cdot 3^2$
   c. $(7 + 3\sqrt{5})^2$
   d. $(94 + 42\sqrt{5})(94 - 42\sqrt{5})$
   e. $94^2 - 5 \cdot 42^2$

Why do we always have to calculate things that are following?

3. **a.** $(2 + \sqrt{5})(2 - \sqrt{5})$
   b. $(2 + \sqrt{5})^2$
   c. $(9 + 4\sqrt{5})(9 - 4\sqrt{5})$
   d. $9^2 - 5 \cdot 4^2$
   e. $(9 + 4\sqrt{5})^3$

   Calculate each of the preceding.

Okay, yeah, that was weird. Let's never do that again.

4. **a.** Build a Box for a Numbers With Friends game targeting $\sqrt{5}$.
   b. Find four different solutions to $y^2 - 5x^2 = 1$ with $x, y > 0$.
   c. How can the ideas from Problem 3 be used to find these solutions?

5. Time for fun with radicals! Find the continued fraction for $\sqrt{39}$.

You can tell your Facebook friends that working with radicals was . . . rad. But don't, because it will fire off some weird rocket ship purple thing.

6. Here is a Box for Numbers With Friends targeting $\sqrt{39}$. And, you just found the continued fraction for $\sqrt{39}$! Woo, time to play Numbers By Yourself!
   a. Fill in the first row of the Box on the next page, then the second row.

b. For each convergent $\frac{y}{x}$, calculate $y^2 - 39x^2$ and write it in the third row.

Numbers By Yourself has vastly grown in popularity. It's even inspired a Billy Idol-sponsored game about dancing.

		# of L moves	# of R moves	# of L moves	# of R moves	# of L moves
		6				
$\frac{0}{1}$	$\frac{1}{0}$					
Start L	Start R	L	R	L	R	L
$y^2 - 39x^2$ value						

Neat Stuff

7. Without using any techmology, find two positive solutions to $y^2 - 23x^2 = 1$ by first doing stuff with $\sqrt{23}$.

8. Pick some integers n that aren't square numbers, and look at each continued fraction for $\sqrt{n}$.

I'm a lasagna hog.

- **a.** Describe a formula for the first coefficient of the continued fraction in terms of n.
- **b.** What is the relationship between the first coefficient of the continued fraction and the last number before the coefficients repeat?
- **c.** Look carefully at the coefficients of the continued fraction *between* the first and the last (before repeat). What do you notice? Huh. Wow! Tacocat??

Go hang a salami!

9. Stephanie explores powers of $1+\sqrt{2}$ and $1-\sqrt{2}$.

- **a.** Make a table of $(1+\sqrt{2})^n$ and $(1-\sqrt{2})^n$ for $n = 0$ through 5.
- **b.** What is true about $(1-\sqrt{2})^n$ as n gets larger?
- **c.** What is true about $(1+\sqrt{2})^n + (1-\sqrt{2})^n$ all the time?
- **d.** Compare $(1+\sqrt{2})^n$ to the numerators of the convergents to $\sqrt{2}$.

What's a Tacocat's favorite Pokémon? Why of course it's Girafarig!

10. If one convergent to $\sqrt{2}$ is $\frac{p}{q}$, find the next convergent in terms of p and q.

11. Pick a different square root and explore Problems 9 and 10. What might you use instead of $(1+\sqrt{2})$?

12. Matthew was inspired by Mia, who in Problem 18 on Day 8, measured how good a fraction is at approximating its target. She multiplied the distance between the fraction and target by its denominator. Matthew goes one step further and multiplies the distance from the target by the *square* of its denominator. If a fraction has a denominator that is 10 times bigger, he says it needs to be 100 times closer. Using Matthew's measure, which fractional approximations are considered the best? Experiment with $\sqrt{5}$, $\sqrt{39}$, $\sqrt{7}$, and any others you've calculated.

If squaring is only one step further, does that mean Mia's method was golden?

13. Jane likes periodic patterns so much that she writes down a continued fraction expansion that's *purely periodic*—the coefficients in the continued fraction repeat and there aren't any coefficients in front that aren't part of the repetition. Her example is

Jane likes periodic patterns so much . . . *How much?* Jane likes periodic patterns so much . . . *How much?* Jane likes . . .

$$4 + \cfrac{1}{9 + \cfrac{1}{5 + \cfrac{1}{4 + \cfrac{1}{9 + \cfrac{1}{5 + \cdots}}}}}$$

a. Determine what number Jane's continued fraction represents.

b. Josie decides to reverse the numbers in Jane's fraction so her coefficients are $5, 9, 4, 5, 9, 4, \ldots$. What number does this represent?

c. Try your own examples of purely periodic continued fractions and figure out a relationship between the numbers corresponding to the original and reversed continued fractions.

Hint: If two expressions of the form $a + b\sqrt{n}$ multiply out in such a way that the $\sqrt{n}$ vanishes, then one expression must be a multiple of the conjugate of the other.

14. **a.** Use a calculator to find the continued fraction for $\sqrt{94}$. Keep going until it repeats... or does it?

b. Compare with `bit.ly/pcmisqrt94`. What?!

I say, WhatsApp with that!

15. Remember keeping track of those streaks in Numbers With Friends? Say we encounter a streak of 100 movements of the left fraction during the game. At some point during the 100 movements, the left fraction be-

Make sure you are playing this game with close friends before streaking.

comes closer to the target number than the right number. Investigate when this happens during each streak for different numbers, such as $\sqrt{41}, \pi, e, \pi \cdot e, \ldots$.

Fun fact: it is *unknown* whether $\pi \cdot e$ is rational or irrational. Wait, that's not fun, that's mostly just scary.

Work through Problems 18–21 from Day 11 before proceeding to Problems 16 through 18.

16. Use induction to prove that B_n always divides evenly into $d-(T_n-a_nB_n)^2$, where $\sqrt{d}$ is the continued fraction we seek.

17. Prove that T_n and B_n are always integers with $1 \leqslant B_n \leqslant d$ and $0 \leqslant T_n \leqslant \sqrt{d}$, where $\sqrt{d}$ is the continued fraction we seek.

What's a Toyota dealer's favorite palindrome? "A Toyota's a Toyota." Wait. Was it a car or a cat I saw? An old cat, one of those senile felines. Ugh, my aching head. And I've just got one lonely Tylenol.

18. Prove that the continued fraction for $\sqrt{d}$ for any positive integer d must repeat. You'll need to make use of the fact that (T_n, B_n) can only take on finitely many different values, and that T_n and B_n determine T_{n+1} and B_{n+1}.

19. Hey, this number is pretty interesting:

$$119333161815121713302033171215181613333911$$

Figure out what's interesting about it. To help you out, the number 30203 is also interesting.

Yes, we find every number interesting.

Tough Stuff

20. Determine the necessary condition(s) for a real number to have a purely periodic continued fraction. Prove your result.

21. Small coefficients seem to be more frequent than large coefficients in continued fraction expansions of real numbers. Determine if there is a distribution that continued fraction coefficients satisfy and find it.

Happy bad e approximation day, everyone! You know what they say about e: never odd or even.

22. Find some lattice points in the coordinate plane that are equidistant from the x-axis and the line $y = x$. Or really close to it.

Day 13: There's Leggy Penguins?

Opener

1. **a.** Build the continued fraction for $\sqrt{14}$ and its first five convergents.
   b. For each convergent $\frac{y}{x}$, calculate $y^2 - 14x^2$.
   c. Find two solutions to $y^2 - 14x^2 = 1$ with $x, y > 0$.

The first two convergents are $\frac{3}{1}$ and $\frac{4}{1}$.

2. Calculate each of these.

   a. $(4 + \sqrt{14})(4 - \sqrt{14})$
   b. $(4 + \sqrt{14})^2$
   c. $(30 + 8\sqrt{14})(30 - 8\sqrt{14})$
   d. $(15 + 4\sqrt{14})(15 - 4\sqrt{14})$
   e. $(15 + 4\sqrt{14})^2$
   f. $(449+120\sqrt{14})(449-120\sqrt{14})$

Important Stuff

3. **a.** Multiply this: $(y + x\sqrt{n})(y - x\sqrt{n}) =$
   b. If that product equals 1, and x and y are positive integers, what can you say about $(y - x\sqrt{n})$?

What's that product equal? No, it's Sweet'N Low.

4. **a.** Elli performs the Euclidean Algorithm (square cutting) to calculate the GCD of 477 and 152.
   b. Mark calculates the continued fraction for $\frac{477}{152}$ in the style of `bit.ly/pcmisqrt7`.
   c. Describe as many connections as you can.

5. Use Matthew's measure (see Problem 12 from Day 12) to determine which rational approximations you've encountered for π are the best.

They're the best . . . around! Nothing's ever gonna keep them down. Never gonna give them up.

6. Instead of measuring the difference between a fraction and a target number, Samantha measures the difference between the *square* of the fraction and the *square* of the target number. Use this idea to think about why integer solutions to $y^2 - nx^2 = 1$ produce such accurate rational approximations of $\sqrt{n}$.

Approximations to $\sqrt{n}$ have the right angle, but approximations to π are truly the best a-round.

Old Stuff

7. In the third century BCE, Archimedes posed a mathematics problem in a letter to Eratosthenes that in-

volved finding the number of white, black, dappled, and brown bulls and cows subject to several arithmetic conditions. (An English translation of that original problem by Hillion and Lenstra is attached.) Revel in the fact that this old problem amounts to finding positive, integer solutions to $y^2 - 410286423278424x^2 = 1$.

We'll be around to check your reveling.

8. Archimedes stated that

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

Do better than Archimedes.

Archimedes was good at math, but we're not sure why he cared so much about dappled things.

9. In the seventh century, Brahmagupta wrote this about the equation $y^2 - 92x^2 = 1$: "A person solving this problem within a year is a mathematician." Patricia challenges you to become a mathematician by finding two solutions to this equation for $x, y > 0$.

10. The Indian mathematician Ramanujan came up with this approximation

$$\pi \approx \left(\frac{2143}{22}\right)^{1/4}$$

which is even more accurate than 355/113, though it requires a root. Calculate the continued fraction for π^4 to reveal how he came up with this approximation.

When Ramanujan had trouble sleeping, he famously counted taxis instead of sheep, and usually didn't fall asleep until counting at least 1729 of them.

Review Your Stuff

11. We traditionally set aside part of the last problem set for review. Work as a group at your table to write **<u>one</u>** review question for tomorrow's problem set. Spend **<u>at most 20 minutes</u>** on this. Make sure your question is something that $\boxed{*\textbf{everyone}*}$ at your table can do, and that you expect $\boxed{*\textbf{everyone}*}$ in the class to be able to do. Problems that connect different ideas we've visited are especially welcome. We reserve the right to use, not use, or edit your questions, depending on how much other material we write, whether you've written your question on the approved piece of paper, your group's ability to write a good joke, how good your bribes are, and hundreds of other factors.

Imagine yourself writing an Important Stuff question, that's what we are looking for here. You got this!

Remember that one time at math camp where you wrote a really bad joke for the problem set? No? Good.

Neat Stuff

12. A *Diophantine equation* is a polynomial equation, usually with two or more unknowns, for which integer solutions are desired. For example, Pythagorean triples are solutions to the Diophantine equation $a^2 + b^2 = c^2$.

a. Find an integer solution to $477x + 152y = 1$.
b. Find integer solutions to $y^2 - 11x^2 = 1$.
c. Use techmology to graph $y^3 - 5x^2 = 1$ and look for integer solutions with $x > 0$.
d. Use techmology to graph $y^2 - 5x^3 = 1$ and look for integer solutions with $x > 0$.

Need a hint? Use your work from Problem 4a.

13. You've now found many solutions to equations of the form $y^2 - nx^2 = 1$. Each solution (x, y) matches a fraction $\frac{y}{x}$ that serves as an excellent approximation to $\sqrt{n}$. But how excellent?

These approximations are most excellent, Ted! Way.

a. $\frac{15}{4} - \sqrt{14} \approx \frac{1}{M}$ for what closest possible integer M?
b. $\frac{9}{4} - \sqrt{5} \approx \frac{1}{M}$ for what closest possible integer M?
c. $\frac{1249}{200} - \sqrt{39} \approx \frac{1}{M}$ for what closest possible integer M?
d. Look for a pattern; if you need to, throw some more fractions at it, but make sure they are solutions to $y^2 - nx^2 = 1$.

14. Use Problems 3 and 13 to show that if (x, y) is a solution to $y^2 - nx^2 = 1$, then $\frac{y}{x} - \sqrt{n}$ is very, very close to $\frac{1}{2xy}$.

Start by exploring the approximate value of $(y - x\sqrt{n})$ in terms of x or y.

15. Revisit Problems 13 and 14 using other convergents, the ones that don't make the magical 1. What changes, what stays the same?

16. Fun with algebra! Rewrite the expression

$$(y_1^2 - nx_1^2)(y_2^2 - nx_2^2)$$

in the form

a. $(y_1y_2 + nx_1x_1)^2 - n(\qquad\qquad\qquad)^2$
b. $(y_1y_2 - nx_1x_1)^2 - n(\qquad\qquad\qquad)^2$

Fun with trivia! The director of *Happy Feet* and *Happy Feet 2* went on to direct his next movie . . . *Mad Max: Fury Road*. He also directed *all* the other Mad Max movies. He also wrote *Babe*, yes that one.

17. Calculators use this recursive formula to generate a decimal approximation for $\sqrt{n}$:

$$x_{m+1} = \frac{1}{2}\left(x_m + \frac{n}{x_m}\right)$$

One reason this works is because x_m and $\frac{n}{x_m}$ are on opposite sides of $\sqrt{n}$, but there are others.

[waves hands frantically, redirecting to Problem 18]

a. Roxanne uses the recursive formula with $n = 2$ and starting value $x_0 = 1$. What happens? Compare to your Box for $\sqrt{2}$.

b. Try $n = 7$ and $x_0 = 3$. What happens? Compare to your box for $\sqrt{7}$.

c. Try other values for n and x_0.

18. Ramona loves the Calculus! She applies Newton's Method to the equation $f(x) = x^2 - n$ to see where the formula in the previous problem comes from.

Newton's Method does not usually involve sitting under a tree and hoping to get hit by stuff.

19. Find some integer solutions for $y^2 - 5x^2 = -1$. Refer back to your Box that you previously built for $\sqrt{5}$, or build one.

20. Have you noticed that small coefficients seem more frequent than large ones in continued fractions? It turns out that for almost every real number, if a_k are the coefficients of its continued fraction, then there is a universal limit to the geometric mean for these coefficients:

$$(a_0 a_1 a_2 \cdots a_{n-1})^{1/n} \to 2.68545... \qquad \text{as } n \to \infty$$

That number is called Khinchin's constant. Try calculating the geometric mean of the first 5, 10, then 15 coefficients of the continued fraction for $\pi + e$. Experiment with other numbers.

This result doesn't work well for irrational solutions to quadratic equations with rational coefficients, because those continued fractions always repeat. But if you pick a positive real number at random, the chance that you'll get one of those is zero.

Tough Stuff

21. For which integers n does the equation $y^2 - nx^2 = -1$ admit integer solutions?

22. The denominators of rational convergents seem to get larger quickly, but they don't grow exponentially. Instead, prove that for almost every continued fraction expansion, the denominators of the convergents, x_n, grow like

$$x_n^{1/n} \to \exp\left(\frac{\pi^2}{12 \ln 2}\right) \qquad \text{as } n \to \infty$$

That limiting value is called Lévy's constant.

And the title of Leggiest Penguin Ever goes to . . . Danny DeVito? Yikes.

From "Solving the Pell Equation" by H. W. Lenstra Jr., *Notices of the American Mathematical Society*, Feb 2002.

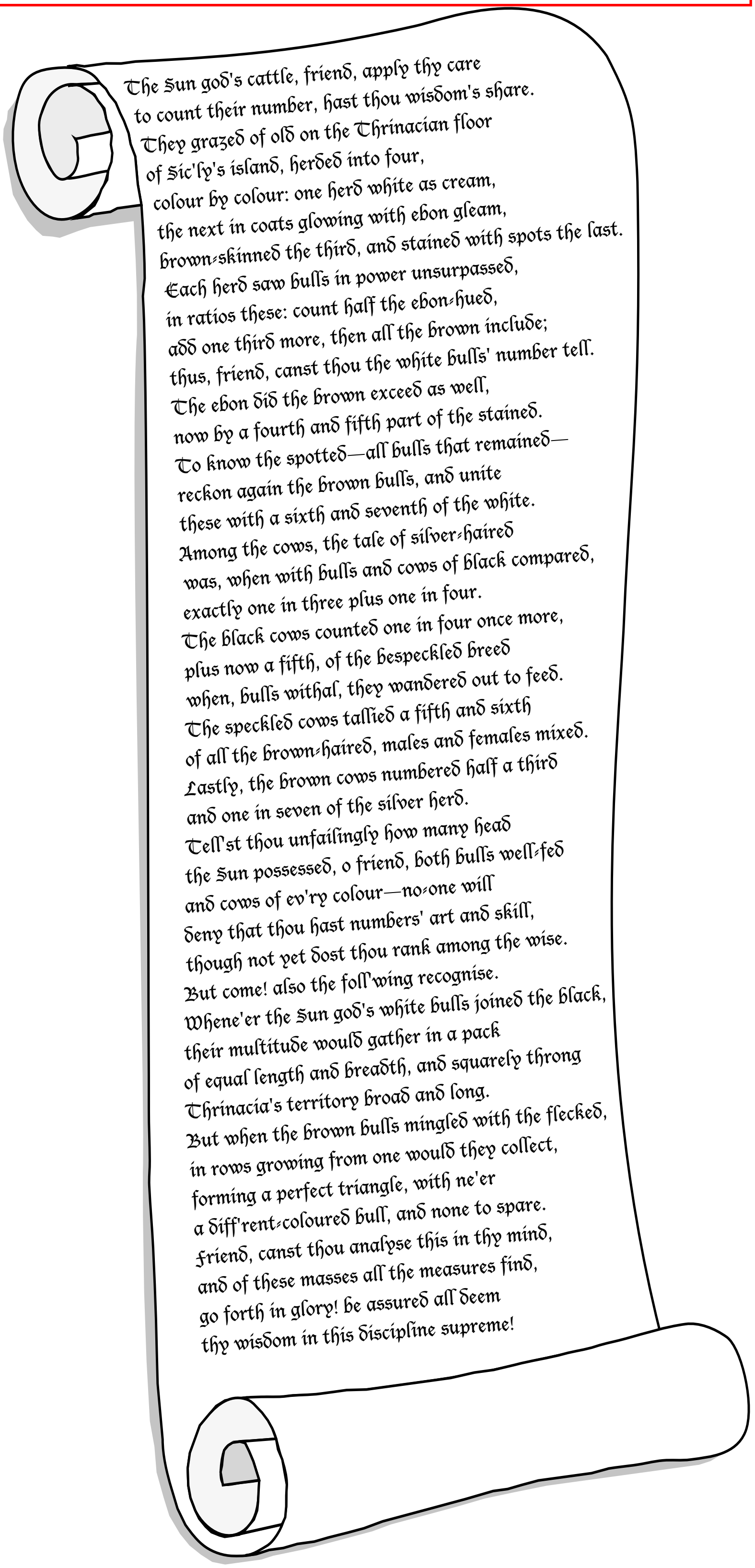


Figure 3. Problem that Archimedes conceived in verse and posed to the specialists at Alexandria in a letter to Eratosthenes of Cyrene.

Writing x, y, z, t for the numbers of white, black, dappled, and brown bulls, respectively, one reads in lines 8-16 the restrictions

$$x = (\tfrac{1}{2} + \tfrac{1}{3})y + t,$$
$$y = (\tfrac{1}{4} + \tfrac{1}{5})z + t,$$
$$z = (\tfrac{1}{6} + \tfrac{1}{7})x + t.$$

Next, for the numbers x', y', z', t' of cows of the same respective colors, the poet requires in lines 17–26

$$x' = (\tfrac{1}{3} + \tfrac{1}{4})(y + y'), \qquad z' = (\tfrac{1}{5} + \tfrac{1}{6})(t + t'),$$
$$y' = (\tfrac{1}{4} + \tfrac{1}{5})(z + z'), \qquad t' = (\tfrac{1}{6} + \tfrac{1}{7})(x + x').$$

Whoever can solve the problem thus far is called merely competent by Archimedes; to win the prize for supreme wisdom, one should also meet the conditions formulated in lines 33–40 that $x + y$ be a *square* and that $z + t$ be a *triangular number*.

The first part of the problem is just linear algebra, and there is indeed a solution in *positive* integers. The general solution to the first three equations is given by $(x, y, z, t) = m \cdot (2226,\ 1602,\ 1580,\ 891)$, where m is a positive integer. The next four equations turn out to be solvable if and only if m is divisible by 4657; with $m = 4657 \cdot k$ one has (x', y', z', t') $= k \cdot (7206360, 4893246, 3515820, 5439213)$.

The true challenge is now to choose k such that $x + y = 4657 \cdot 3828 \cdot k$ is a square and $z + t$ $= 4657 \cdot 2471 \cdot k$ is a triangular number. From the prime factorization $4657 \cdot 3828 = 2^2 \cdot 3 \cdot 11 \cdot$ $\cdot 29 \cdot 4657$ one sees that the first condition is equivalent to $k = al^2$, where $a = 3 \cdot 11 \cdot 29 \cdot 4657$ and l is an integer. Since $z + t$ is a triangular number if and only if $8(z + t) + 1$ is a square, we are led to the equation $h^2 = 8(z + t) + 1 = 8 \cdot 4657 \cdot 2471 \cdot al^2 + 1$, which is the Pell equation $h^2 = dl^2 + 1$ for

$$d = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 \cdot (2 \cdot 4657)^2$$
$$= 410\,286\,423\,278\,424.$$

Thus, by Lagrange's theorem, the cattle problem admits infinitely many solutions.

In 1867 the otherwise unknown German mathematician C. F. Meyer set out to solve the equation by the continued fraction method [3, p. 344]. After 240 steps in the continued fraction expansion for $\sqrt{d}$ he had still not detected the period, and he gave up. He may have been a little impatient; it was later discovered that the period length equals 203254. The first to solve the cattle problem in a satisfactory way was A. Amthor in 1880 (see [6]). Amthor did *not* directly apply the continued fraction method; what he did do we shall discuss below. Nor did he spell out the decimal digits of the fundamental solution to the Pell equation or the corresponding solution of the cattle problem. He did show that, in the smallest solution to the cattle problem, the total number of cattle is given by a number of 206545 digits; of the four leading digits 7766 that he gave, the fourth was wrong, due to the use of insufficiently precise logarithms.

Day 14: Please Choose My Iguana

Opener

1. **a.** What is the fraction with the smallest denominator that is between $\frac{27}{73}$ and $\frac{10}{27}$?

   b. What is the area of the parallelogram contained in the Alpha Quadrant with vertices $(0,0)$, $(73,27)$, and $(27,10)$?

   c. Use Pick's Theorem (Opener from Day 7) to explain why there can't be any lattice points inside that parallelogram. Then, explain why your answer to part a has the smallest denominator.

We talkin' bout *fractions?*

Oh right, there's totally one more vertex.

Darryl gets really upset when someone says "vertice" for the singular form of vertices.

Important Stuff

2. Beth and Will play Numbers With Friends with a target of $\sqrt{5}$. Beth starts at $\frac{0}{1}$ and Will starts at $\frac{1}{0}$.

   a. Beth moves first. How many times, and to what number?

   b. Will moves second. How many times, and to what number?

   c. What happens next? And then?

Or we talkin' bout *game*?

After today, there truly will be no and then.

3. A rectangle has dimensions $\sqrt{5}$ by 1. Jake performs Westley-style square cutting on this rectangle. Little does he know, it's a trap!

   a. How many 1-by-1 squares can Jake cut?

   b. How many of the next level of squares are cut? Hm I guess their side lengths are $(\sqrt{5}-2)$.

   c. Keep going for a while! Notice anything?

This rectangle was built by Admiral Ackbar himself.

Jake's got time for a long adventure . . .

4. Amy fires a laser at a 45-degree angle from the bottom left corner of a room! Sadly, this room is $\sqrt{5}$ by 1.

   a. What's unfortunate about that?

   b. As the laser travels, Amy counts the number of times it hits one of the top or bottom walls, and separately counts the number of times it hits one of the left or right walls. She notices that sometimes, these hits are very close to one another, closer than ever before. When?

It's a trap!

5. David notices that when he looks at convergents to $\sqrt{n}$ and calculates $y^2 - nx^2$, the value is always negative, then positive, then negative, etc. Why is this true? Remembering Numbers With Friends should help!

Left shark never does get those moves right.

6. We've learned that if n isn't a perfect square, then some convergent $\frac{y}{x}$ to $\sqrt{n}$ is a solution to $y^2 - nx^2 = 1$. But which one, Dylan asks?

Definitely not Peggy.

a. The continued fraction for $\sqrt{39}$ has a repeating period of 2 in its coefficients: 6, 4, 12, 4, 12 . . . Which convergent is the first one to solve $y^2 - 39x^2 = 1$? The second?

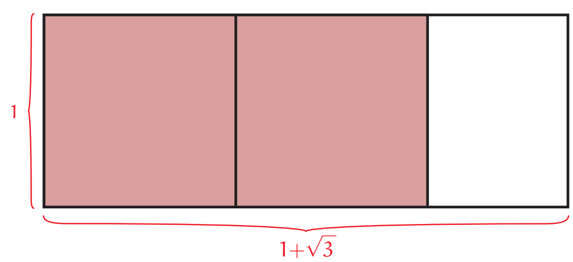
Yes, that's right, the second! Wait, was that the question?

b. The continued fraction for $\sqrt{23}$ has a repeating period of 4 in its coefficients: 4, 1, 3, 1, 8, 1, 3, 1, 8 . . . Which convergent is the first one to solve $y^2 - 23x^2 = 1$? The second?

c. The continued fraction for $\sqrt{41}$ has a repeating period of 3 in its coefficients: 6, 2, 2, 12, 2, 2, 12 . . . Which convergent is the first one to solve $y^2 - 41x^2 = 1$? What happened, David?

What happened? Well, okay . . . Our whole universe was in a hot, dense state . . .

7. Here's a $(1+\sqrt{3})$-by-1 rectangle.



When you went through the square cutting procedure in Problem 3, you probably noticed it got increasingly difficult to figure out how many boxes fit when both dimensions of the remaining rectangle involve radicals. Mary offers a modified version of the square-cutting algorithm: each time you remove the greatest number of squares leaving behind a rectangle, rotate that rectangle 90° counterclockwise, then scale the rectangle so that the height of the box is 1 again. Try a few rounds of this for the rectangle above. How does it connect to `bit.ly/sqrt3plus1`?

Never trust a rectangle with a big butt and a smile.

Never trust a thick glaze in a pile. That sauce is hoisin!

Your Stuff (Choose Your Own Adventure)

T1. Given a n-by-$n+1$ rectangle, try Jennifer's style of cutting. Jennifer tires quickly and doesn't want to cut out as many squares. In any n-by-$n+1$ rectangle, divide the even side by 2 and cut 2 squares of that size. Then, keep cutting squares according to Westley's style of cutting until you finish.

Investigate different n by $n+1$ rectangles and compare the number of squares you can cut compared to if you purely used Westley's style of cutting.

Your jokes. *Our jokes.*

Someday we'll find it, the Westley connection, the cutting, the fractions, and ϕ.

T2. **a.** Let n be a perfect square bigger than 3. Find the pattern for the continued fractions for $\sqrt{n+2}$ and $\sqrt{n-2}$.

b. Use the patterns in part a to predict the continued fractions for $\sqrt{51}$ and $\sqrt{47}$.

So, don't pick 49 in part a, I guess!

T3. Write the first five coefficients of the continued fraction for π. Play Numbers With Friends and create the Box for the game up to five columns.

		# of L moves	# of R moves	# of L moves	# of R moves	# of L moves
$\frac{0}{1}$	$\frac{1}{0}$					
Start L	Start R	L	R	L	R	L

What happens if you eat a continued fraction portion of the π? (That'd be hard to rationalize.)

T4. $\frac{244}{63}$ is an OK approximation for $\sqrt{15}$. Find a better approximation as many different ways as you can.

Approximation scampi, approximation gumbo, approximation creole, pineapple approximation, coconut approximation, approximation soup, approximation stew, approximation and potatoes, approximation sandwich. That's about it.

T5. **a.** Find some solutions to $y^2 - 10x^2 = 1$ with $x, y > 0$.

b. Find multiple convergents of $\sqrt{17}$.

c. Play Numbers With Friends targeting $\sqrt{26}$.

d. Describe as many connections as you can.

e. Predict the continued fractions of $\sqrt{37}, \sqrt{50}, \ldots$.

T6. Given

$$x = 6 + \cfrac{1}{6 + \cfrac{1}{6 + \cdots}}$$

a. Find x.

b. Write the continued fractions for $2x$, $3x$, and $4x$.

c. Can you generalize to nx?

I just saw it, I think it was on the last page.

One day you'll know, how far it'll go. #teammary

T7. **a.** Use the Euclidean Algorithm to divide a 6-by-7 rectangle.

b. Find another division of the 6-by-7 rectangle that uses a smaller number of squares than you found in a, using any method.

c. Use the Euclidean Algorithm to divide a 6-by-8 rectangle. Repeat part b for this rectangle.

d. Split the following work among the members of your team. Represent the fractions $\frac{7}{6}$ and $\frac{8}{6}$ in each of the following ways. Make as many connections as you can as a group and use your work in the previous parts.

- Graph (lattice points)
- Continued fraction
- Box/Mega Box
- Bonus: Verbal description

Don't stop believing there's some major connection here . . . but then again, don't spend too much time on this!

T8. Kevin only likes to eat 1 inch by 1 inch squares of bacon. Daniel only eats 2 in by 2 in squares of bacon as he washes it down with a Mountain Dew Big Gulp. Mary likes to take 3 in by 3 in bacon squares on her morning job because they're big on protein. Sean likes to do a morning bacon mask that is 8 in by 8 in square. a Fresh Market sells cut-to-size bacon rectangles that come directly from the mountain tops of the Wasatch range where Westley the Butcher runs his shop.

Who's Kevin? His favorite temperature is 6°F.

Etymology bakkon "back meat."

In Shakespeare, Bacon is a divisive term for "a rustic."

a. Find the dimensions of the rectangle given Westley the Butcher square cutting if everyone gets at least one of their desired square sizes.

b. They start fighting because they all want the same number of pieces. Is this possible?

c. What is the least number of straight slices Westley must make to satisfy everyone in part a?

Westley the Butcher, I think he was the big bad from Dexter Season 5. I kind of stopped watching by then.

T9. Bowen told Darryl that he could only generate any lattice point using the convergents of $\sqrt{41}$ without using any techmology. Bowen lists $(2, 13)$, $(62, 397)$, and

As usual, Darryl is perfect. This is Darryl's fight song. Take back his life song! Bowen's got fight for his right to GRAAAAAAPH!

$(320, 2049)$ as evidence! As usual, Darryl wants to prove Bowen wrong. For which points, if any, can Darryl prove Bowen wrong. Explain why and show how.

Don't worry, they'll bond later by watching sprinklers water the hell out of a fuse box near the 7-Eleven.

T10. **a.** Find five positive values of m that make

$$\sqrt{m+\sqrt{m+\sqrt{m+\sqrt{m+\cdots}}}}$$

a positive integer.

b. What patterns do you notice?

c. Why were Matthew, Mark, Mary, and Michelle placed at the same table? Where are Luke and John?

John Luke ended up on Duck Dynasty, yo.

More Stuff

8. On a n-by-n chessboard, place n coyotes in such a way that they leave the maximum number of squares for sheep to safely occupy. Coyotes can attack sheep in other spaces of the board like a queen can in chess.

a. Let u_n be the maximum number of squares that can be "safe" on a n-by-n board. Determine u_n for various values of n.

b. Show that $u_n/n^2 \to \alpha$ for some constant α as $n \to \infty$.

c. Calculate the continued fraction for α and determine which u_n/n^2 are convergents for α.

What's this α? Is this why people don't fall asleep counting sheep, because they skipped over α?

9. Let $\frac{y_n}{x_n}$ be the nth convergent of the continued fraction expansion of the real number α. Prove that

$$\left|\frac{y_n}{x_n}-\alpha\right| < \frac{1}{x_n x_{n+1}} \leqslant \frac{1}{x_n^2}$$

Yo, convergents bouncing up and down, makes me want to know what done this! Up down up, feel the vibrations! Come on, come on.

10. Kristen and Kristen, rebellious as usual, decide that they are tired of always starting their Boxes with $0/1$ and $1/0$. They want to start their Boxes with any two fractions they want. They fill in the top row of their Box (next page) with the continued fraction coefficients for $\sqrt{2}$, then they fill in the second row using the usual method (see `bit.ly/ksbox`).

The rebellion was caused by an Empire State of Mind. And Peggy.

This usually also involves shouting Bismillah a few times, even though it's obvious he will not let you go.

		a_0	a_1	a_2	a_3	a_4	
		1	2	2	2	2	
$\frac{\alpha}{\beta}$	$\frac{\delta}{\gamma}$				$\dfrac{12\alpha+17\delta}{12\beta+17\gamma}$		

What fractions will they get in the second row and what will those fractions converge to? Hint: Write

$$\frac{12\alpha+17\delta}{12\beta+17\gamma} = \frac{\alpha+\frac{17}{12}\delta}{\beta+\frac{17}{12}\gamma}$$

Goodness gracious, great balls of Fireball under pressure by David Bowie and Dancing Queen!

Continued Fraction Arithmetic Stuff

11. Come up with an algorithm for deciding which of two continued fractions is bigger.

12. Come up with an algorithm for finding the reciprocal of a continued fraction.

13. Say you know the continued fraction for $\sqrt{2}$, but now you want the continued fraction for $2\sqrt{2}$. The Kristens have done something quite useful for you! (Please do Problem 10 first.)

"You know the continued fraction for $\sqrt{2}$."

If they choose 0/2 and 1/0 as their starting fractions, and keep the same coefficients for $\sqrt{2}$, then they would have ended up with $\sqrt{2}/2$. Therefore, to find the continued fraction coefficients for $2\sqrt{2}$, use 0/2 and 1/0 as the starting fractions and find new coefficients that maintain the same convergents as for $\sqrt{2}$ with $\frac{0}{1}$ and $\frac{1}{0}$. Try that in the AwesomeBox below.

Everything is AwesomeBox! Everything is coolbox if you're part of a teambox.

		a_0	a_1	a_2	a_3	a_4	
		1	2	2	2	2	
$\frac{0}{1}$	$\frac{1}{0}$						
$\frac{0}{2}$	$\frac{1}{0}$						
		2					
		b_0	b_1	b_2	b_3	b_4	

14. Repeat the method of the previous problem to find the continued fraction for $2\sqrt{7}$, given that for $\sqrt{7}$. Hmmm!

You'll be a Magic Box hero, got $\sqrt{7}$ in your eyes.

		a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	
		2	1	1	1	4	1	1	1	4	
$\frac{0}{1}$	$\frac{1}{0}$										
$\frac{0}{2}$	$\frac{1}{0}$										
		5	3		3						
		b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	

15. Use the method above to calculate the continued fraction for $2\sqrt{3}$ given the continued fraction for $\sqrt{3}$. OMG!

Mary loves OMG, they made that song "If You Leave". Oh, crap, you're all leaving.

16. Read `https://perl.plover.com/classes/cftalk/INFO/gosper.txt` and please explain it to us. Thanks!

No More Stuff

17. Thanks. We had a wonderful time and hope you did too. See you again as soon as possible. What's next for you?

We miss you so, so bad.