Day 1: Shuf’ling

Welcome to PCMI. We know you’ll learn a lot of mathematics here—maybe some new tricks, maybe some new perspectives on things with which you’re already familiar. A few things you should know about how the class is organized.

- **Don’t worry about answering all the questions.** If you’re answering every question, we haven’t written the problem sets correctly.

Some of the problems have yet to be solved. Those are the *really* fun ones.

- **Don’t worry about getting to a certain problem number.** Some participants have been known to spend the entire session working on one problem (and perhaps a few of its extensions or consequences).
- **Stop and smell the roses.** Getting the correct answer to a question is not a be-all and end-all in this course. How does the question relate to others you’ve encountered? How do others think about this question?
- **Be excellent to each other.** Believe that you have something to learn from everyone else. Remember that everyone works at a different pace. Give everyone equal opportunity to express themselves. Don’t be afraid to ask questions.
- **Teach only if you have to.** You may feel the temptation to teach others in your group. Fight it! We don’t mean you should ignore your classmates but give everyone the chance to discover. If you think it’s a good time to teach your colleagues about the orbit stabilizer theorem, think again: the problems should lead to the appropriate mathematics rather than requiring it.
- **Each day has its Stuff.** There are problem categories: Important Stuff, Neat Stuff, Tough Stuff. Check out the Opener and the Important Stuff first. All the mathematics that is central to the course can be found and developed there. *That’s* why it’s Important Stuff. Everything else is just neat or tough. Each problem set is based on what happened the day before.

When you get to Day 3, come back and read this introduction again.

Will you remember? Maybe . . .

Opener

1. Let's watch a video. Don't worry, it's only like 2 minutes long.

Wait *what?* Discuss this with your tablemates and try to figure out what you can about this. After 15 minutes, move on to Problem 2.

What is this I don't even.

Important Stuff

2. Does the perfect shuffle work for other deck sizes? If not, why not? If so, what stays the same and what changes? Distribute the work among your tablemates and discuss.

3. Melanie is thinking of a positive integer, and because she's a math teacher she calls it x. What information would you know about x based on each statement?

- **a.** $3x$ has last digit 4
- **b.** $7x$ has last digit 4
- **c.** $4x$ has last digit 4
- **d.** $5x$ has last digit 4

4. **a.** What number is $9 \cdot 10^1 + 9 \cdot 10^0 + 4 \cdot 10^{-1} + 4 \cdot 10^{-2}$?

b. Noel's favorite number is 802.11_{10}. Write it as a sum of powers of 10.

The little 10 here means the number is in base 10. Bases will generally be given when they seem needed, and we'll try not to be confusing.

5. **a.** What number is $1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 0 \cdot 3^0 + 1 \cdot 3^{-1}$?

b. Brittany's favorite base-3 number is 2110.2_3. Write it as a sum of powers of 3.

c. Convert 2110.2_3 to base 10.

Surely you came to PCMI armed with your favorite number in each base.

6. Write each number as a decimal. Write each number as a decimal. Write each number as a decimal.

These directions either terminate or repeat.

- **a.** $\frac{1}{2}$
- **b.** $\frac{1}{50}$
- **c.** $\frac{1}{9}$
- **d.** $\frac{2}{9}$
- **e.** $\frac{9}{9}$
- **f.** $\frac{1}{13}$

7. Hey, we just met you, and this is crazy; but here's some numbers, so make them base three.

We missed you so bad. We missed you so, so bad. Also an appropriate Canadian reference! It's always a good time.

a. 9

b. 13

c. 242

d. $\frac{1}{9}$

e. $\frac{1}{13}$

f. $\frac{1}{242}$

8. Write each number as a base-3 "decimal".

a. $\frac{1}{9}$

b. $\frac{1}{2}$

c. $\frac{2}{2}$

d. $\frac{3}{2}$

e. $\frac{1}{13}$

f. $\frac{1}{242}$

Neat Stuff

9. Under what circumstances will a base-10 decimal repeat?

Perhaps it wasn't heard the first time.

10. Under what circumstances will a base-3 decimal repeat?

11. The repeating decimal $.\overline{002}$ means .002002002 . . . . But what number is it? That depends on the *base*! Ace this problem by finding the base-10 fraction equal to $.\overline{002}$ in each given base.

Psst: You did some work on base 3 already. Ace this base problem, and you'll see the sine.

a. base 3

b. base 4

c. base 5

d. base 7

e. base n

f. base 2?!

12. We overheard Amari and Grace debating about whether or not the number .99999 . . . was equal to 1. What do you think? Come up with a convincing argument, and if you already know one, come up with a different one!

We didn't hear who was arguing each side, we mostly just ran away.

13. **a.** Find all positive integers n so that the base-10 decimal expansion of $\frac{1}{n}$ repeats in 3 digits or less.

b. Find all positive integers n so that the base-3 "decimal" expansion $\frac{1}{n}$ repeats in 4 digits or less.

14. Patrick has a cube, and he wants to color its faces with no more than two different colors. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.

Two of the colorings are possible using PAAS Dip & Dye.

15. Justin has randomized two four-card decks with the same cards: ace, two, three, four. He will flip over the top card of each deck, hoping the same card is in the same position for both decks. If not, he'll flip the next top card from each deck, still hoping.

a. What is the exact probability that Justin will be disappointed?

b. This gives a 1 in N chance of disappointment. Write N to nine decimal places.

16. Write 13 and 242 in base $\sqrt{3}$ instead of base 3. Hee hee hee. Or maybe this turns out to be totally awesome!

Tough Stuff

17. Peg has randomized two thirteen-card decks with the same cards: ace, two, three, four, . . . , jack, queen, king. She will flip over the top card of each deck, hoping the same card is in the same position for both decks. If not, she'll flip the next top card from each deck, still hoping.

This gives a 1 in P chance of disappointment. Write P to nine decimal places.

P is for Peg, and that's good enough for me.

18. Lauren has a cube, and she wants to color its *edges* with two different colors. What now!

19. **a.** Convert 13 to base $\frac{3}{2}$.

b. Convert 13 to base π.