Day 5: Super Bowl

Opener

1. Complete this table. Don't worry, none of it is in base 2. Long division is fun! While it's a good idea to split the work among one another, please don't use any technology in this work or you may miss some big ideas.

Fraction	Decimal representation	# of repeating digits
1/3	$0.\overline{333}$	1
1/5	0.2	n/a
1/7	$0.\overline{142857}$	6
1/9		
1/11		
1/13		
1/15		
1/17		
1/19		
1/21		
1/23	$0.\overline{0434782608695652173913}$	22
1/25		
1/27		

We are the PCMI Shufflin' Crew. Shufflin' on down, doin' it for you.

Jokes so bad we know we're good. Blowin' your mind like we knew we would.

You know we're just shufflin' for fun, struttin' our stuff for everyone.

Rock paper scissors for who gets stuck doing $1/19$. Or volunteer, ya masochist.

Important Stuff

2. a. When doing long division, how can you tell when a decimal representation is about to terminate?
   b. How can you tell when a decimal representation is about to repeat?

Digits come, one by one, 'til you shout, "Enough, I'm done!" Then we'll give you even more while you protest . . .

3. Explain why the decimal representation of $\frac{1}{n}$ can't have more than n repeating digits.

Is there a better upper bound than n digits?

4. Find all solutions to each equation in mod 10.

a. $0x = 1$
b. $1x = 1$
c. $2x = 1$
d. $3x = 1$
e. $4x = 1$
f. $5x = 1$
g. $6x = 1$
h. $7x = 1$
i. $8x = 1$
j. $9x = 1$

In *mod 10*, the only numbers are 0 through 9. For example, $6 + 5 = 1$ and $6 \cdot 5 = 0$. Refrigerator Perry's number changes to 2, but McMahon gets to keep his 9.

5. **a.** Complete this multiplication table for mod 8.

$\times$	0	1	2	3	4	5	6	7
0								
1								
2				6				
3	0							
4								
5								
6			4					
7								1

In *mod 8*, Fridge would be especially unhappy to see his number changed to 0, since he'd be stuck wearing the same number as the punter with the cowbell and Panama hat.

b. How many 1s do you see in your table above?

Don't include the ones on the sidelines.

6. Numbers can multiply with other numbers to make 1! It happens, but not always. Whenever this happens, both numbers are called *units*.

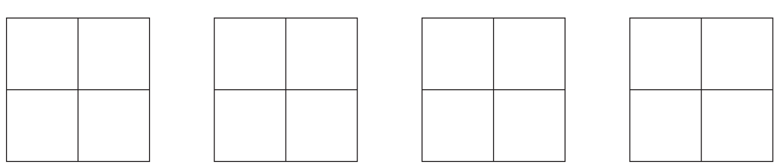
a. If you're working just with integers, what numbers are units?
b. If you're in mod 10, what numbers are units?
c. List all the units in mod 8.
d. List all the units in mod 15.

When working with integers only, 5 is not a unit, since $5 \times \frac{1}{5} = 1$. But . . .

The only time 0 is a unit is in an Algebra 2 book.

7. Here are four uncolored copies of Spencer's 2×2 chessboard. Each uncolored chessboard has four rotational symmetries: you can rotate it by $0°, 90°, 180°, 270°$ about the center and it will look identical.

Rotation by $0°$ *really* doesn't do anything to the chessboard, or any other object, so it's called the *identity transformation.*



This 2×2 chessboard also has four reflective symmetries. Describe the four reflective symmetries of the chessboard by drawing their lines of symmetry above.

8. Joshua and Kim each bought one of Spencer's 2×2 chessboards and painted each of the four squares with their favorite color(s).

They can choose any colors, perhaps the Bears' black and white, or perhaps the hapless Patriots' red and blue. Or even just one color!

- **a.** Joshua says his chessboard has 180° rotational symmetry (about the center). What must be true about the colors of his four squares?
- **b.** Kim says her chessboard has 90° rotational symmetry. What must be true about the colors of her four squares?
- **c.** How many reflective symmetries could Joshua's colored chessboard have?

Neat Stuff

9. Pick some more mods. Try to determine rules for what numbers are units in mod m, and how many units there are. Keep picking more mods until you have a feel for it.

"A Feel For Units" was narrowly rejected as the title for Chaka Khan's greatest hit.

10. Use `http://go.edc.org/15cards` to keep track of the position of each card in a 15-card deck as it undergoes perfect shuffles. Complete this table.

Card No.	Positions	Cycle Length	Card No.	Positions	Cycle Length
0	$0, 0, 0\dots$	1	8		
1			9		
2			10		
3	$3, 6, 12, 9, 3, \dots$	4	11		
4			12	$12, 9, 3, 6, 12, \dots$	4
5			13		
6			14		
7					

11.
- **a.** In Problem 10, card numbers 3 and 12 were essentially on the same cycle. How many unique cycles does the 15-card deck have?
- **b.** There are 15 cycle lengths in Problem 10. Add up the *reciprocals* of all 15 numbers. What up with that? Look for an explanation.

I say, oooo weee, what up with that, what up with that?

12. What size decks will get restored to their original order after exactly 10 perfect shuffles (and not in any fewer number of shuffles)?

I'm Samurai Mike I stop'em cold. Part of the defense, big and bold.

I've been jammin' for quite a while, doin' what's right and settin' the style.

Give me a chance, I'll rock you good, nobody messin' in my neighborhood.

(This man went on to coach the 49ers.)

13. **a.** How many units are there in mod 9? Call this number "Liz".

b. Build a multiplication table for mod 9 *but only include the units*. This multiplication table's size will be Liz-by-Liz.

c. Use `http://go.edc.org/9cards` to track the positions of all cards in a 9-card deck as they undergo perfect shuffles. Hmmm?

Fun Fact: Da Bears were *nominated for a Grammy award* for their performance, and they remain the only professional sports team with a Top 41 hit single. (Wait, there's a Top 41 now?)

14. How many unique cycles did you observe when tracking the locations of cards in the 9-card deck?

15. Can a power of 2 be a multiple of 13? Explain.

16. **a.** Multiply out these expressions and explain what they tell you about $2^n - 1$ when n has factors.

$$(2^a - 1)(1 + 2^a + 2^{2a} + 2^{3a} + \cdots + 2^{(b-1)a}) = ?$$
$$(2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + \cdots + 2^{(a-1)b}) = ?$$

b. Find the three prime factors of $2^{14} - 1$ astoundingly quickly, by hand.

17. What size decks will get restored to their original order after exactly 14 perfect shuffles (and not in any fewer number of shuffles)?

I'm mama's boy Otis, one of a kind. The ladies all love me for my body and my mind.

I'm slick on the floor as I can be, but ain't no sucker gonna get past me.

18. Determine all deck sizes that can be restored to their original order in 15 or fewer perfect shuffles, and the specific number of shuffles needed for each.

19. Abby, Rob, Suzanne, and Trisha are waiting in line, wondering if they can get to any of their 24 possible arrangements through these two rules:

- The person in the back of the group may jump to the front: TARS ⇒ STAR
- The person in the third position may switch with the person in the first position: TARS ⇒ RATS

Can all 24 possible arrangements be made? Make a graph illustrating the options.

Oh, RATS!

Who decides this? Perhaps it is up to the TSAR. Afterwards, they will head to the museum of fine ARTS and then get some well-deserved RAST. What, that's SART of a word.

20. *It's p-adic number time!* Here are two interesting 10-adic numbers, and we're only going to show you their last six digits. (There are more digits to the left, feel free to try and figure out what they are.)

$$x = ...109376.$$

$$y = ...890625.$$

a. Calculate $x + y$.
b. Calculate the product xy.
c. Calculate x^2 and y^2.
d. How *crazy* are the p-adic numbers?!

Hey I'm Bowen, from EDC. I write books called CME.

I stay up late most every night, writin' problems that come out right.

I add numbers fast, just like magic, but my hairline has gotten tragic.

You all got here on the double, so let's all do the PCMI shuffle . . .

Willie Nelson? Patsy Cline? Aerosmith? Britney Spears? Queen Bey? Gnarls Barkley? Eddie? Horse? Frog? Queen? Train? Madonna is crazy for $U(n)$.

21. Find a mathematical equation that is true in mod 2 and mod 3, but not true in general.

22. Find a mathematical equation that is true in mod 2, mod 3, mod 4, and mod 5, but not true in general.

Psst: you can skip mod 2. Why?

Tough Stuff

23. That thing this morning. How'd we do that?

How'd we do what? You know what. The thing, with the thing.

24. Let n be an integer. Let $U(n)$ be the set of all deck sizes that are restored to its original order after exactly, and no fewer than, n perfect shuffles. (You can disregard trivial decks with 0 or 1 cards.) In Problem 12, you calculated $U(10)$.

a. Prove that $U(n)$ is never empty: there is always some deck for which n shuffles is the lowest possible number.
b. For which n does $U(n)$ contain only one element?

25. **a.** For $n = 1$ through $n = 7$, find all n-digit numbers who last n digits match the original number. For example, $25^2 = 625$, ending in 25.
b. Find a connection between this and the p-adic numbers, or decide that there is no such connection.

My name's Darryl, I'm from LA. I work with math most ev'ry day.

I love to laugh, I love to eat, my Mathematica programs can't be beat.

I've taught kids of every age, now I'm in Park City writin' page by page.

I'm not here to fuss or fumble, I'm just here to do the PCMI Shuffle . . .