Day 6: Cupid

Opener

1. **a.** Find the repeating decimal for $\frac{1}{41}$. List all the remainders you encounter during the long division, starting with 1 and 10.

   b. Write $\frac{100}{41}$ as a mixed number.

   c. Find the repeating decimal for $\frac{18}{41}$. List all the remainders you encounter, including 1 and 10.

   d. Find the repeating decimal for $\frac{1}{37}$. List them remainders!

   e. Find the repeating decimal for $\frac{1}{27}$. Coolio.

No, it's not Coolio, the guy's name is Cupid. He even named the dance after himself, which seems a little presumptuous.

2. Complete this table. Splitting up the work is a great idea, but please do it without fancy spreadsheets or computer programs.

You're going to have to walk it by yourself, now walk it by yourself.

n	Powers of 10 in mod n	Cycle Length
41		
37		
3		
7		
9		
11		
13	$1, 10, 9, 12, 3, 4, 1, \ldots$	6
17		
19		
21		
23	$1, 10, 8, 11, 18, 19, 6, 14, 2, 20, 16,$ $22, 13, 15, 12, 5, 4, 17, 9, 21, 3, 7, 1, \ldots$	22
27		

→ Reminder: You don't have to calculate 10^5 to figure out the 4 in this row. Since you already know the previous 3 is equal to 10^4 in mod 13, you can use $3 \times 10 = 30 = 4$ mod 13. It's super helpful! Also, be careful of Cupid's arrows. This arrow points to the right, to the right.

Important Stuff

3. **a.** What are the rotational symmetries of a regular pentagon? Of a regular hexagon?

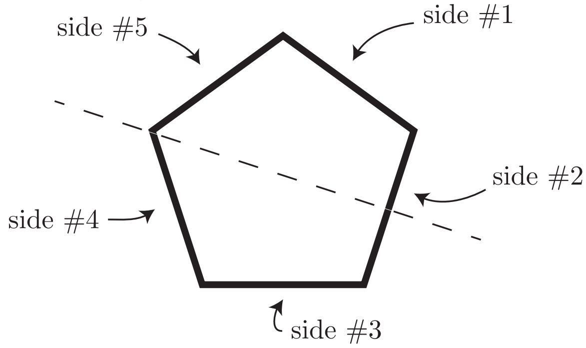
   b. What are the reflective symmetries of a regular pentagon? Of a regular hexagon?

   c. How many rotational symmetries does a regular n-gon have? How many reflective symmetries?

Don't forget the *identity transformation*, which is a rotation by $0°$.

4. **a.** Jeanette bends a paperclip into a regular pentagon. She has 37 colors and paints each of the five sides with some color(s). If her colored paperclip has this reflective symmetry shown below, what must be true about the colors of the five sides? How many possible paint jobs are there?

One option is to paint all five sides in Snugglepuss, and another is to paint four sides Snugglepuss and one side puce.



b. Hannah bends a paperclip into a regular hexagon. She also has 37 colors and paints each of the six sides with some color(s). If her colored paperclip has 120° rotational symmetry, what must be true about the colors of the six sides? How many possible paint jobs are there?

5. **a.** What do these equations have to do with rewriting the number 227 as 1402_5 (in base 5)?

The screen has a color-coded version of this problem. The notorious RGB!

$$\begin{aligned} 227 &= 45 \cdot 5 + 2 \\ 45 &= 9 \cdot 5 + 0 \\ 9 &= 1 \cdot 5 + 4 \\ 1 &= 0 \cdot 5 + 1 \end{aligned}$$

Explain what's happening in each equation above.

b. Here are some equations to help you rewrite 227 in base 3. Fill in the missing numbers.

This must be some sort of spin-off from the original 227. Perhaps an entire show called Jackée about a character not named Jackée.

$$\begin{aligned} 227 &= \quad \cdot 3 + 2 \\ 75 &= 25 \cdot 3 + \\ 25 &= \quad \cdot 3 + 1 \\ &= 2 \cdot 3 + 2 \\ &= 0 \cdot 3 + \end{aligned}$$

c. Rewrite 227 in base 7.

Another 227 spin-off, 22/7, was the first show to prominently feature an approximation of pi. It lasted almost a week.

. . . sorry, I'm being told there was a one-character typo in the top note.

6. Complete this table. A number x is a *unit* in mod n if there is a number y such that $xy = 1$. In Day 5, many people noticed that this is also the list of numbers in mod n that have no common factors with n.

You can say x is *relatively prime* to n, which is totally different than saying that x is *optimus prime*.

n	Units in mod n	# units in mod n
8	$1, 3, 5, 7$	4
15	$1, 2, 4, 7, 8, 11, 13, 14$	8
25		
30		
49	*too many units*	

← Was that just a *Sneakers* reference? I guess it is now! To the left, to the left.

7. Find the numbers n in the middle column of Problem 2's table that are . . .

a. . . . factors of 9.

b. . . . factors of 99 but not of 9.

c. . . . factors of 999 but not of 9 or 99.

d. . . . factors of 9999 but not of 9, 99, or 999.

e. . . . factors of 99999 but not of 9, 99, 999, or 9999.

f. . . . factors of 999999 but not of . . . alright already.

What's up with that?

Ferris has been absent 9 times. 9 times? NINE TIMES.

I got 99 factored, and 7 ain't one.

Appropriate Beatles song: Number Nine.

Appropriate Nine Inch Nails song: 999999.

Ooooo weeeee, what up with that, what up with that!

8. Calculate each of the following. You may also want to look back at Day 5.

a. $999999 \div 7$
b. $999999 \div 13$
c. $99999 \div 41$
d. $999999 \div 37$

⇒ Only five 9s this time! Cupid's arrow still points to the right, to the right.

Neat Stuff

9. How many units are there in mod 105? Counting them all would be a little painful.

10. Collin points out that the list of units in mod 15 contains 2 and 7, and $2 \times 7 = 14$ is also a unit.

a. Solve $2x = 1$ and $7y = 1$ in mod 15.
b. For x and y above, what is $14xy$ in mod 15?
c. Explain why, if x and y are units in mod 15, then xy is also a unit.

Bonus option: solve part b before solving part a!

11. Lauren points out that the list of units in mod 15 includes some powers of 2: $2, 4, 8, 1$.

a. Write a complete list of all the powers of 2 in mod 15. OK!

b. Explain why, if x is a unit in mod 15, then x^2 is also a unit.

c. Same for x^p for any positive integer power p.

d. Why won't there just be billions of units if you can take any unit to *any* power p and make another one?

OK, Cupid? OKCupid's website says "We use math to get you dates"; its founder has a math degree. The founder probably knows that these powers of 2 can open Jacuzzi doors.

12. What size decks will get restored to their original order after exactly 10 perfect shuffles (and not in any fewer number of shuffles)?

13. For what n does the decimal expansion of $\frac{1}{n}$ have an immediate repeating cycle of 10 digits and no fewer?

For example, $\frac{1}{15} = 0.0\overline{6}$ does *not* immediately repeat.

14. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be an ordered list in mod 9.

a. Find a number M such that $2M = 1$ in mod 9.

b. Calculate $M \cdot S$ in mod 9. *Keep everything in its original order.*

c. Calculate M^2S in mod 9.

d. Calculate M^kS in mod 9 for all k until something magical happens, then look back to see just how magical it is.

$\Leftarrow$ M^2S can also be used to send pictures to someone's phone. For large k, M^kS indicates that several people enjoyed a meal. To the left, to the left.

15. **a.** What are the units in mod 11?

b. Which of the units in mod 11 are perfect squares? Which are not perfect squares?

c. What are the units in mod 13?

d. Which of the units in mod 13 are perfect squares? Which are not perfect squares?

In mod 11, 5 is a perfect square because $4 \cdot 4 = 5$. It's not the same in mod 13!

16. Euler conjectured that it takes at least k kth powers to add up to another one. For example, $3^2 + 4^2 = 5^2$ but you need three cubes to add up to another cube. In the 1960s this was finally disproven:

$$133^5 + 110^5 + 84^5 + 27^5 = n^5$$

Without a calculator, and hopefully without multiplying it all out, find the value of n.

You fool, Euler! You fool!

17. **a.** There are 16 units in mod 17. Of them, make a list of the eight that are perfect squares and the eight that are not.

b. If x and y are perfect squares, is xy a perfect square... always? sometimes? never?

c. If x is a perfect square and y isn't, is xy a perfect square... always? sometimes? never?

d. If neither x nor y is a perfect square, is xy a perfect square... always? sometimes? never?

I get stupid. I shoot an arrow like Cupid. I'll use a word that don't mean nothin', like looptid. Hey, that guy named the dance after himself, too!

18. There are values of k for which there is only one deck size that restores in k perfect shuffles and no fewer. Are there values of k for which there is only one denominator n such that $\frac{1}{n}$ has a k-digit repeating decimal?

19. A *repunit* is a number made up of all ones: 11111 is a repunit. Investigate the prime factorization of repunits, and determine the values of k for which the k-digit repunit is prime.

20. Hey, what's up with skipping $n = 15$ in Problem 2?

a. What is the length of the repeating portion of the decimal representation of $\frac{1}{15}$?

b. What is the smallest positive integer n for which $10^n = 1$ in mod 15?

c. Aren't your answers for the previous two problems supposed to match? Figure out what's going on and rectify the situation.

Tough Stuff

21. **a.** Expand $(x-1)(x-2)(x-3)\cdots(x-6)$ in mod 7.

b. Calculate $1^6, 2^6, 3^6, \ldots, 6^6$ in mod 7.

c. For p prime and $x \neq 0$, prove $x^{p-1} = 1$ in mod p.

Turns out lots of people get to name dances after themselves, including Dougie, Hammer, Urkel, Freddie, Macarena, Batman, Bartman, those little doll kids, Pee-Wee, Ben Richards, Bernie from Weekend at Bernie's, Chubby Checker, apparently Ketchup, and, of course, Carlton.

22. You can move the top card in a 52-card deck to any position, with a lot less than 52 shuffles. You just need a sequence of both perfect shuffles and Patricia-style shuffles where the right-side card is placed first (see Problem 10 from Day 3). Find a method for moving the top card to any desired position in the deck with six or fewer shuffles.