Day 7: iPod

Opener

1. Use the handout from Day 3 to help you fill out this table row-by-row.

n	$2^n - 1$	Factors of $2^n - 1$ (less than 100)	Deck sizes requiring exactly n perfect shuffles (less than 100)
1	1	1	1, 2
2			
3	7	1, 7	7, 8
4			
5			
6			
7			
8	255	1, 3, 5, 15, 17, 51, 85	17, 18, 51, 52, 85, 86
9			
10	1023	1, 3, 11, 31, 33, 93	11, 12, 33, 34, 93, 94
11			

Unlike other tables, this one really needs to be done row-by-row, by everyone. Don't split this work among your tablemates, but do check with one another!

Important Stuff

2. Working with your table, fill in a whole lot of *this* table:

`http://www.tinyurl.com/numberofunits`

A number x is a *unit* in mod n if there is a number y such that $xy = 1$. This is also the list of numbers in mod n that have no common factors with n.

Any table caught violating the instructions in the spreadsheet will be forced to listen to the Super Bowl Shuffle on infinite repeat.

3. What do these equations have to do with $\frac{1}{13}$?

$$\begin{aligned} 1 &= 0 \cdot 13 + 1 \\ 10 &= 0 \cdot 13 + 10 \\ 100 &= 7 \cdot 13 + 9 \\ 90 &= 6 \cdot 13 + 12 \\ 120 &= 9 \cdot 13 + \\ &= 2 \cdot 13 + \\ &= \quad \cdot 13 + \end{aligned}$$

Finish the equations, then use the same method to find the decimal expansions of $\frac{2}{13}$ and $\frac{1}{77}$.

Oops, we didn't finish it all! For a colorful version, see the online class notes. If you're already looking at the online class notes, please stop reading this sentence... now. (Good.)

Free Slurpee Day is coming . . . very soon.

4. **a.** List the powers of 10 in mod 77.
b. How long is the repeating decimal for $\frac{1}{77}$?
c. What is $999999 \div 77$?
d. Explain why the length of the repeating decimal for $\frac{1}{n}$ is the same as the length of the cycle of powers of 10 in mod n.

I'm at the Pizza Hut, I'm at the Taco Bell. I'm at the Taco Bell, I'm at the Pizza Hut. I'm at the permutation Pizza Hut and Taco Bell!

5. Complete this table with help from techmology. Wooo!

n	# units in mod n	$2^{(\text{\# units in mod } n)}$ in mod n	$10^{(\text{\# units in mod } n)}$ in mod n
13			
17			
21			
41			
51			
77			

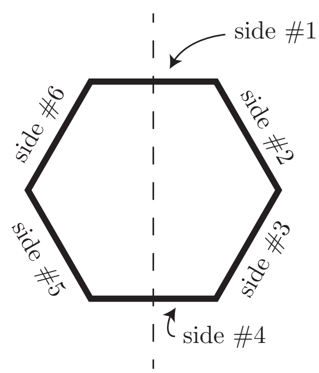
We recommend using `wolframalpha.com` instead of your calculators. For example, try typing `10^6 mod 7` into the box.

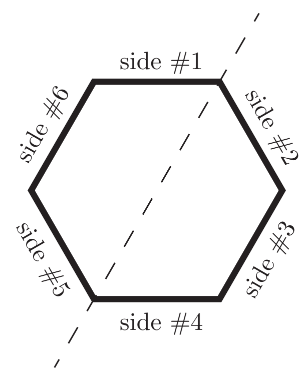
What implications does this table have for shuffling cards and repeating decimals?

Yo. Science, what is it all about. Techmology. What is that all about? Is it good, or is it wack?

6. Maria and Marissa both bend a paperclip into a regular hexagon. Both of them paint each of the six sides of their paperclip from a palette of 227 colors.

Repeated colors are allowed. Next year's course will mostly be about the art of paper clip folding, which is usually called borigami.





a. Maria wants her paperclip to have the reflective symmetry shown on the left above. How many possible ways can her paperclip be painted?
b. Marissa wants her paperclip to have the reflective symmetry shown on the right above. How many possible ways can her paperclip be painted?

Yo. Writing out a number like 2,655,237,841 is probably not necessary.

Neat Stuff

7. Suppose you know that in mod n, there's an integer $x \neq 1$ that makes $x^{10} = 1$.

- **a.** Find some other integers k for which you are *completely sure* that $x^k = 1$.
- **b.** Find some integers $k < 10$ for which it is *possible* that $x^k = 1$.
- **c.** Find some integers $k < 10$ for which it is *definitely impossible* that $x^k = 1$.

8.
- **a.** The decimal expansion of $\frac{1}{7}$ is $.\overline{142857}$. Use this to find the decimal expansions of all $\frac{n}{7}$ for $0 \leqslant n \leqslant 7$.
- **b.** The decimal expansion of $\frac{1}{13}$ is found in Problem 3. Use this to find the decimal expansions of all $\frac{n}{13}$ for $0 \leqslant n \leqslant 13$.
- **c.** What is the same and what is different about the set of expansions of $\frac{n}{7}$ and $\frac{n}{13}$?

9. Edward performs perfect shuffles on a 12-card deck. Write out the order of his cards after a few shuffles.

0	1	2	3	4	5	6	7	8	9	10	11

After Edward completes the shuffles, he sparkles! Also, he hates werewolves, and Jacob, and has a team.

10. Let $H = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ in mod 11.

- **a.** Find a number A such that $2A = 1$ in mod 11.
- **b.** Calculate $A \cdot H$ in mod 11. *Keep everything in its original order.*
- **c.** Calculate A^2H in mod 11.
- **d.** Calculate A^kH in mod 11 for all k until something interesting happens. You decide what counts as interesting!

A^2H indicates that you're at the dentist after too many snacks. For large k, A^kH indicates that zombies are nearby.

11. Lila suggests you look for connections between Problems 9 and 10.

12. Use `http://go.edc.org/27cards` to figure out how many unique cycles are made when a 27-card deck undergoes perfect shuffles. (Refer back to Problem 10 on Day 5.) Can you find a more efficient way to count the number of unique cycles?

Sheesh, nobody told us that *every day* we'd be shuffling. Oh, they did? Still, it seemed like such an empty threat.

13. **a.** There are 6 units in mod 7. For each number x in mod 7, compute x^6 in mod 7. Cool!

Cool. Cool cool cool.

b. There are 4 units in mod 8. For each number x in mod 8, compute x^4 in mod 8. Cool?

NOT COOL!!

14. Suppose you know that in mod n, $ab = 1$ and $cd = 1$.

a. Explain why $a, b, c,$ and d are all units in mod n.

b. Find something that solves this equation:

This proves that the product of two units is a unit.

$$(ac) \cdot (\qquad) = 1$$

c. Find something that solves this equation:

This proves that the power of a unit is a unit, and the daughter of a Zappa is also a unit.

$$(a^k) \cdot (\qquad) = 1$$

15. **a.** Write out all the units in mod 15.

b. Pick one of them and call it v. Now multiply v by all the units in mod 15 (including itself). What numbers do you get?

c. If the units are called $u_1, u_2, u_3, \dots, u_8$ show that

$$u_1 \cdot u_2 \cdot u_3 \cdots u_8 = (v \cdot u_1) \cdot (v \cdot u_2) \cdot (v \cdot u_3) \cdots (v \cdot u_8)$$

d. Show that $v^8 = 1$ in mod 15.

16. The decimal expansion of $\frac{1}{7}$ is $0.\overline{142857}$. Now, split the repeating digits in half and add them together:

$$142 + 857 = 999.$$

Try this with other fractions $\frac{1}{n}$ that have an even number of repeating digits. Any ideas why this works? What about some other splits?

Some examples include $\frac{1}{13}$, $\frac{1}{73}$, $\frac{1}{91}$, and for the adventurous, $\frac{1}{17}$ and $\frac{1}{23}$.

17. One way some people counted up the units in mod 105 was this: *I'll take all 105 numbers, then subtract the number of multiples of 3, then 5, then 7. Oh, wait, then I have to add back in the multiples of 15, 21, and 35. Oh, wait, then I have to subtract the multiples of 105.*

a. Does this work? There are 48 units in mod 105.

b. What's the probability that a number picked between 1 and 105 *isn't* a multiple of 3? 5? 7?

c. Consider the product $(3-1)(5-1)(7-1)$.

Yes, and . . . ??

18. Change the argument a little from Problem 15 to show that for any unit in any mod,

$$u^{(\text{\# units in mod } n)} = 1 \text{ in mod } n$$

19. Nathan watches 6♠ (spades) as a 52-card deck undergoes perfect shuffles. Before each shuffle, he notes whether the 6♠ is in the left half (aka the top half) or the right half of the deck. Write a 0 if the card is in the left half, and a 1 if it is in the right half. Cupid says you should do it again with 10♢, and there is room for you to pick your own card to try.

See the animations at `tinyurl.com/pcmi52cards`. I heard the ace of spades is cosmic.

To the right, to the right . . . come on, you know the Cupid Shuffle! And so does 10♢, apparently!

Card	Which half just before shuffle #?							
	1	2	3	4	5	6	7	8
6♠								
10♢								

a. For each card, compute the base 2 number that corresponds to the eight 0s and 1s you listed.

These numbers run from 0 to 255.

b. Given a card, is there a way to construct its sequences without watching the shuffles?

Since you asked, probably, yes?

20. Abraham hands you his favorite multiple of 5, written as an eight-digit base 2 number: $011010??_2$. Oh noes, you can't make out the last two digits. What must those missing digits be for the number to be a multiple of 5?

Tough Stuff

21. Consider different odd primes p and q. The number p may or may not be a perfect square in mod q, and the number q may or may not be a perfect square in mod p. Seek and find a relationship between these two things! Respek.

There is so little respek left in the world, that if you look the word up in the dictionary, you'll find that it has been taken out.

22. Consider different odd primes p and q. The length of the repeating decimal of $\frac{1}{p}$ in base q may be odd or even, and the length of the repeating decimal of $\frac{1}{q}$ in base p may be odd or even. Seek and find a relationship between these two things! Booyakasha.