Day 8: Slurpee Mix

Opener

1. What are the powers of 10 in mod 21, and how many are there?

2. Find each decimal expansion in base 10. Try to complete the table with the least amount of computation! For example, Christian suggests that you fill in 10/21 first.

If you use a calculator, report your answers as repeating decimals instead of rounding them off.

Fraction	Decimal	Fraction	Decimal
1/21	$0.\overline{047619}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21		20/21	

Hey, me just met you, and this is crazy, but you got Slurpee, so share it maybe?

Important Stuff

3. **a.** How many fractions in the table above are in lowest terms?

b. How many units are there in mod 21?

c. Pick three different units in mod 21. For each, calculate u^{12} in mod 21.

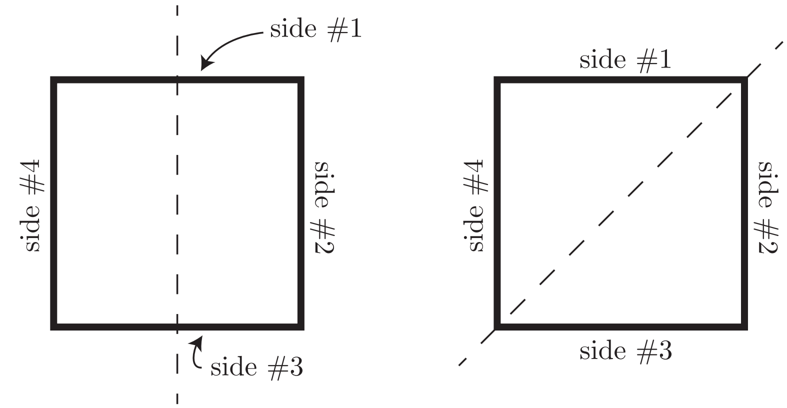
7/11 is in lowest terms, but the ripoff copy store 6/12 is not.

4. Describe any patterns you notice in the table above. What fractions form "cycles" that use the same numbers in the same order? Write out each cycle in order. How many unique cycles are there? How long are the cycles?

Coicles? I don't see any coicles here, nyuk nyuk nyuk.

5. Gabe and Gary both bend a paperclip into a square. Both of them paint each of the four sides of their paperclip from a palette of k colors.

While you're welcome to call these Square One and Square Two, only one of them can host Mathnet.



a. Gabe wants his paperclip to have the reflective symmetry shown on the left above. How many possible ways can his paperclip be painted?

b. Gary wants his paperclip to have the reflective symmetry shown on the right above. How many possible ways can his paperclip be painted?

Your answers here will be expressions with k in them. This problem has squares and k because Circle K doesn't sell Slurpees.

6. How do these equations relate to the base 2 "decimal" for $\frac{1}{21}$?

$$\begin{aligned} 1 &= 0 \cdot 21 + 1 \\ 2 &= 0 \cdot 21 + 2 \\ 4 &= 0 \cdot 21 + \\ 8 &= \quad \cdot 21 + 8 \\ 16 &= 0 \cdot 21 + \\ 32 &= 1 \cdot 21 + \\ 22 &= 1 \cdot 21 + \end{aligned}$$

Finish the equations above.

As per use, we have a nice version in fabulous color on screen. While it is possible to get a Slurpee with red, blue, and green flavors, we don't recommend drinking it.

7. What are the powers of 2 in mod 21, and how many are there?

8. Let's try the opener again, but this time we'll use *base 2*. Find each "decimal" expansion. Try to complete the table with the least amount of computation. For example, Monika suggests that you fill in 2/21 first.

If you use a calculator, good luck finding a calculator that reports answers in base 2! You'd be better off using a deck of cards.

Fraction	Base 2 "Decimal"	Fraction	Base 2 "Decimal"
1/21	$0.\overline{000011}$	11/21	
2/21		12/21	
3/21		13/21	
4/21		14/21	
5/21		15/21	
6/21		16/21	
7/21		17/21	
8/21		18/21	
9/21		19/21	
10/21	$0.\overline{011110}$	20/21	

From Cookie Monster's "Share It Maybe": Me got a wish on my mind, it is a chocolate chip kind. Me look at you and me tell, you might have snickerdoodle. Me start to really freak out, please someone call a Girl Scout. Don't mean to grumble or grouse, this taking toll on my house. Me going off my rocker, please give me Betty Crocker.

9. Here is a 21-card deck being perfectly shuffled.

`http://go.edc.org/21cards`

Follow some cards and follow some remainders. How can you use the cards to find the *entire base 2 expansion* for each fraction?

Wait, . . . we *are* better off using a deck of cards? Ohh my.

Neat Stuff

10. **a.** Write out all the units in mod 21.

b. Write out all the powers of 10 in mod 21 (again), starting with 1 and 10.

c. Take all the powers of 10 in mod 21 and multiply them by 2. What happens?

d. Take all the powers of 10 in mod 21 and multiply them by 3. What happens?

e. Take all the powers of 10 in mod 21 and multiply them by 7. What happens?

Hey, you're still in mod 21! There's no such number as 32.

11. Complete this table.

n	"Decimal" for $1/n$ in base 2	# of repeating digits	# of units in mod n
3	$0.\overline{01}$	2	2
5			
7			
9			
11			
13			
15			
17			
19	$0.\overline{000011010111100101}$	18	
21	$0.\overline{000011}$	6	

Using the method from Problem 6 may be helpful to your sanity, but you can also use long division. Split the work, but keep away from techmology or you may miss some big ideas.

12. Cut out the 2×2 chessboards, one for each person in your group. The uncolored chessboard has four rotational symmetries ($0°$, $90°$, $180°$, and $270°$ counterclockwise) and four reflective symmetries (—, ╲, |, and ╱).

a. Maria tells you to rotate your chessboard by $90°$ counterclockwise, then to reflect it (flip it over) along the — (horizontal) axis. Those two operations are equivalent to a single reflection. Which reflection is it?

b. Marila tells you to reflect your chessboard along the — (horizontal) axis, then rotate it by $90°$ counterclockwise. Those two operations are equivalent to a single reflection. Which reflection is it?

Though the chessboard is uncolored, they are labeled so that you can keep track of which square is which. The four squares are labeled like the four quadrants of the coordinate plane.

These moves were popularized by Daft Punk in their hit song "Turn It Flip It, Flip It Turn It". This is not an invitation to turn the square one more time.

13. To fill in each space in the table on the next page, first take your 2×2 chessboard and perform the transformation matching the column of the space, then do the transformation matching the row. Fill in the space with the transformation that is equivalent to the composition of those two transformations.

·	Rot 0°	Rot 90°	Rot 180°	Rot 270°	Refl —	Refl ╲	Refl ∣	Refl ╱
Rot 0°								
Rot 90°		Rot 180°						
Rot 180°						Refl ╱		
Rot 270°								
Refl —								
Refl ╲					Rot 270°			
Refl ∣								
Refl ╱								

14. Compare and contrast Problem 13 with Problem 5 from Day 5.

I noticed this table was wider. I wonder how an even wider table might look on this page.

15. The table you built in Problem 8 has a pile of repeating base 2 "decimal" parts. But what about the numbers *within* the repeating digits?

- **a.** The base 2 "decimal" for $\frac{10}{21}$ is $0.\overline{011110}$. What base 10 number is made from those repeating digits?
- **b.** Do this a few more times for other n until you can explain how to write the base 2 "decimal" for any $\frac{n}{21}$ directly.

16. OK, so about that magic trick we've done once or twice. How'd we do that? The work in Problem 15 is helpful, but you'll need to make some serious modifications. The work in Problems 19 and 20 on Set 7 is also helpful.

17.
- **a.** If you haven't yet, go back and do Problem 15 from Set 7.
- **b.** The *order* of a unit u in mod n is the smallest power $k > 0$ such that $u^k = 1$ in mod n. Prove that the order of any unit u in mod n must be a factor of the number of units in mod n.

You've got to go back Marty! Back to . . . oh, the past. Bah.

Anarchy in the u^k!

18. The decimal expansion of $\dfrac{1}{7}$ is $0.\overline{142857}$. Now, split the repeating digits in thirds and add them together:

$$14 + 28 + 57 = 99.$$

Try this with other fractions $\frac{1}{n}$ whose repeating digit lengths are multiples of 3. What's up with that!

Slurpee sales on July 11 are typically about 40% *higher* than normal, even though they are also being given away for free. Sometimes, if your timing is bad, Slurpee sales are also soupy sales.

19. So, shuffling. Madhu is so good at it that she can split a deck of cards into three equally-sized piles and perfect-shuffle them.

You try it too. Pick a number of cards that is a multiple of three and perform her perfect triple shuffles. See what you find. We recommend numbering your cards as 0, 1, 2, ... There's a lot to find!

Just ask her! Okay maybe we are lying. But we told the truth about that Cookie Monster song, maybe.

Curly: I'm tryin' to think, but nothin' happens!

20. Suppose a and b are relatively prime. How does the length of the decimal expansion of $\frac{1}{ab}$ compare to the lengths of the decimal expansions of $\frac{1}{a}$ and $\frac{1}{b}$?

21. In a Reader Reflection in the *Mathematics Teacher* (March 1997), Walt Levisee reports on a nine-year-old student David Cole who conjectured that if the period of the base-10 expansion of $\frac{1}{n}$ is $n - 1$, then n is prime. Prove David's conjecture, and discuss whether it might apply to other bases.

Warning: whatever Slurpee flavor you pick, do not pick the "Mystery Flavor". Stores take whatever flavor is not selling at all, and relabel it as "Mystery Flavor" hoping to sell some more of it. So, heed this warning, unless you want your Slurpee to taste like Crunch Berries. (You don't.)

Tough Stuff

22. The Fibonacci numbers start with $F(0) = 0, F(1) = 1, F(2) = 1$, etc:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Show that if m is a factor of n, then $F(m)$ is a factor of $F(n)$. For example, $F(7) = 13$ is a factor of $F(14) = 377$.

23. Show that if F is a Fibonacci number greater than 2, then neither $F^2 - 1$ nor $F^2 + 1$ is prime.

24. **a.** Prove that if p is an odd prime, there is at least one base $b < p$ in which the expansion of $\frac{1}{p}$ has period $p - 1$.

b. Determine the number of such bases in terms of p.