Day 11: The Rotary

Opener

1. Kevin has a bunch of congruent non-square rectangles. Because he colors each vertex of each rectangle either corn or cornflower blue, he says that there are 16 different colorings. However, some colorings look identical to others through a re-orientation. Determine the number of "unique" colorings of the rectangles up to re-orientation.

We encourage you to use the index cards to try this out at your tables. When you color the corners of your cards, remember to color both sides.

Important Stuff

2. Complete Handout #1.

a. For each rectangle, what can you say about its orbit size, compared to its number of symmetries?

b. Add up the numbers in the third row. What happens? Can you explain why this works?

c. What do you notice about the numbers in the fourth row?

d. The sum of numbers in the fourth row is $\frac{\boxed{}}{4}$.

Aw jeez, how many handouts this time? Wouldn't you like to know!

Hey that's not a question. That's just a weird box. WHAT'S IN THE BOX???

3. Kate says she can calculate the number of unique colorings of an object up to re-orientation by counting all the symmetries among all possible colorings, then dividing by something.

a. What. Does this even work?

b. What is the thing she divides by?

c. Test this theory on one of the handouts from Day 10.

Do not test the theory with string, and definitely do not test the theory with big bangs.

4. **a.** Complete Handout #2.

b. How does Row 2 of Handout #1 relate to the bottom row of Handout #2?

What up with that, I say, what up with that?

5. Brian loves Kevin's rectangles, but feels stifled by only having two colors. He wants a palette of 31 colors.

a. If we are unconcerned with re-orientations, how many possible different colorings are there?

Stifled, shuffled, it's all in the mind . . .

Hm, this number might be copyrighted. BR has one of the best logos ever. They don't sell cabinets or frappes, but they will add jimmies for no charge.

b. Now Brian tells you that he is only interested in colorings with 180° rotational symmetry, though it's okay if those colorings have other symmetries too. How many possible colorings are there now, being unconcerned about re-orientations?

Hey, don't forget to use the bubbler at break today!

6. Look at the Ultimate Table. Find an important relationship between the number of units in mod n and the cycle length of 2^k in mod n that is consistently true. Do it again between the number of units in mod n and the cycle length of 10^k in mod n.

Hey wait a minute, this is just Handout #3 in disguise!

Here are some unimportant relationships between the columns:

"The columns all contain numbers."

"The columns all contain positive numbers."

"The columns all contain one-, two-, and three-digit numbers."

"None of the entries in any column is the number 8675309."

Neat Stuff

7. Describe what Handout #2 might have looked like if there were 31 colors instead of 2 colors: How many rows and columns? How many check marks in each row?

8. Zarina points out that even though Brian has a palette of 31 colors, many of the 31^4 different colorings look identical to others through a re-orientation. Determine the number of "unique" colorings of these rectangles up to re-orientation.

9. a. Patrick has a cube and paints each face either peach or pink. How many unique colorings are possible? By "unique" we mean that you can't make one look like the other through a re-orientation.

b. Megan also has a cube, but has a palette of three colors to choose from for the faces, instead of two. Estimate how many "unique" colorings there are.

This is also Problem 14 from Day 1. This is also Problem 11 from Day 10. This is also Problem 9 from Day 11. Choosing every face pink or peach is allowed.

This is difficult, so estimate! Don't spend as much time here as you might waiting for a tasty mango dessert.

10. Angela wants to corner the made-to-order market for bracelets with two equispaced beads. Determine the number of unique bracelet designs if she allows for 2 colors. 3 colors? 4 colors? k colors?

11. Nancy's made-to-order bracelet company offers three bead colors to choose from. Watch out, Rob! How many unique bracelet designs can be made if each bracelet has 4 equispaced beads? Five beads?

Equispaced also refers to the positioning of horses at the start of a race.

12. Three Amandas describe three different ways to find multiple rotational symmetries of a cube. Follow along, specifying the axis of rotation and the angle for each symmetry..

a. Amanda tells you to hold the cube with your fingers at the centers of two opposite faces and spin!

b. Amanda tells you to hold the cube with your fingers at two furthest-apart vertices and spin!

c. Amanda tells you to hold the cube with your fingers at the midpoints of two parallel and opposite edges and spin!

d. The descriptions by the three Amandas can help you find all 24 symmetries of the cube. How many symmetries does each Amanda describe?

There are 24 rotational symmetries of the cube, including the identity transformation.

Wanna see this? Go to `go.edc.org/gleamingcube`.

Lookit dem cubes! They're gleaming! Christian Slater would be proud, but I heard he turned into a robot.

These edges are catty-corner from one another.

13. Build an addition table for $\mathbb{Z}_4$. Oh, that's just mod 4. Also, build a multiplication table for $\mathbb{U}_5$. Remember $\mathbb{U}_5$ contains all the units in mod 5, which are 1, 2, 4 and 3. What's with the weird order? That's word.

$(\mathbb{Z}_4, +)$	0	1	2	3
0				
1				
2				
3				

$(\mathbb{U}_5, \times)$	1	2	4	3
1				
2				
4				
3				

What word? Also what's with New Englanders using "rotary" and "catty-corner" and "jimmies"? It's enough to make you want to head to the packie, then sit down with the clicker while eating a fluffernutter and drinking some tonic. Go Sawx.

14. **a.** In $\mathbb{U}_9$ there are six numbers: 1, 2, 4, 5, 7, 8. The number 2 is called a *generator* of $\mathbb{U}_9$ because every number is a power of 2: $1, 2, 4, 8, 7, 5, 1, \ldots$. Which other numbers in $\mathbb{U}_9$ are generators?

b. What are the generators in $\mathbb{Z}_6$ under addition? Instead of using powers (repeated multiplication) of numbers, you will need to use repeated addition.

c. How can generators help you match the tables for $(\mathbb{U}_9, \times)$ and $(\mathbb{Z}_6, +)$?

It tours around $\mathbb{U}_9$, from 1 then all the way
It's 2, go 2, generate it now 2
That's all there is to say
$\mathbb{U}$ can't touch this

15. Do Problem 13 from Day 10 if you haven't already, then compare to the addition table for $\mathbb{Z}_6$. Can you make those two tables match by pairing the entries in some way? Explain!

16. What are the generators in $\mathbb{Z}_7$ under addition? In $\mathbb{Z}_8$? Neat.

Of course it's neat, this is Neat Stuff.

17. Describe some conditions under which you can say that the tables for two operations can *definitely not* be matched. Find more than one condition!

I told you, homeboy, $\mathbb{U}$ can't match this!

18. Make a table of the number of generators of $\mathbb{U}_n$ for different n. What patterns do you notice?

$\mathbb{U}$ need to calm down.

19. **a.** Find the cycle lengths of 3 and 5 under multiplication in $\mathbb{U}_{13}$ and explain why each is *not* a generator.
b. What are the generators of $\mathbb{U}_{13}$?

Never mind, I'll find someone like $\mathbb{U}$ . . .

20. **a.** Find the number of perfect shuffles that it will take to restore a deck of 90 cards. Do this without a calculator and using an efficient method, given what you learned in Problem 6.
b. Find the number of repeating digits in the decimal for 1/73 without long division or a calculator.

$\mathbb{U}$ might be more efficient if $\mathbb{U}$ look back at Problem 20 from Problem Set 10.

I guess the change in my pocket wasn't enough, I'm like, forget $\mathbb{U}$.

21. Make a multiplication chart for all 24 rotational symmetries of the cube! Mwahahaha!

This problem is quite a grinder! I'm not sure if it has any meatballs, though.

22. Find a number n so that the fraction $\frac{1}{n}$ is a repeating decimal with exactly 8,675,309 digits.

Tough Stuff

23. If p is prime, prove that every $\mathbb{U}_p$ has at least one generator.

thank $\mathbb{U}$, next.

24. If p is prime, prove that every $\mathbb{U}_p$ has exactly ________ generators. Hm, looks like we left that spot blank.

25. Find all *composite* n for which $\mathbb{U}_n$ has generators.

Who are $\mathbb{U}$? Who who, who who?

26. For p prime, find general conditions under which the number 2 is *definitely* or *definitely not* a generator in $\mathbb{U}_p$.

If you missed any of today's comments, look them up on $\mathbb{U}_{\text{tube}}$.